V -SUPER VERTEX IN-ANTIMAGIC TOTAL LABELINGS OF DIGRAPHS

G. Marimuthu¹, M.S. Raja Durga*, G. Durga Devi

Department of Mathematics, The Madura College, Madurai- 625011, Tamilnadu, India.
yellowmuthu@yahoo.com, msrajadurga17@gmail.com, punithadurga1991@gmail.com


¹ Corresponding author.

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Abstract

Let $D = (V, A)$ be a directed graph with $p$ vertices and $q$ arcs. For $v \in V$, let $I(v) = \{u \in V | (u, v) \in A\}$. A vertex in-antimagic total labeling is a bijection $f : V(D) \cup A(D) \rightarrow \{1, 2, 3, \ldots, p + q\}$ with the property that
\[
\left\{ f(v) + \sum_{u \in I(v)} f((u, v)) : v \in V(D) \right\}
\]
consists of distinct integers. Such a labeling is called a $V$-super vertex in-antimagic total labeling ($V$-SVIAMT labeling) if $f(V(D)) = \{1, 2, 3, \ldots, p\}$. A digraph $D$ is called a $V$-super vertex in-antimagic total digraph ($V$-SVIAMT digraph) if $D$ admits a $V$-SVIAMT labeling. In this paper, we introduce and study the existence of such labelings for some classes of digraphs. In particular, we investigate the existence of $V$-SVIAMT labeling of generalized de Bruijn digraphs which are used in interconnection network topologies.

1 Background

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. By a digraph $D = (V, A)$ we mean a finite digraph which may possibly admit self-loops but not multiple arcs. The order and the size of a graph or a digraph are denoted by $p$ and $q$ respectively.

For standard graph theory terminology we follow [3, 12]. Let $D = (V, A)$ be a digraph with vertex set $V$ and arc set $A$. If $a = (u, v) \in A$, then $a$ is said to join $u$ to $v$; $u$ is the tail of $a$, and $v$ is its head. If $(u, u) \in A$, the arc $(u, u)$ is called a self-loop or loop. For a vertex $v \in V$, the sets $O(v) = \{u | (v, u) \in A\}$ and $I(v) = \{u | (u, v) \in A\}$ are called the out-neighbourhood and the in-neighbourhood of the vertex $v$, respectively. The out-degree and in-degree of $v$ are $\deg^+(v) = |O(v)|$ and $\deg^-(v) = |I(v)|$, respectively.

A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to the positive or nonnegative integers). The most common choices of domain are the set of all vertices and edges (such labelings are called total labelings), the vertex-set alone (vertex-labelings), or the edge-set alone (edge-labelings). Other domains are possible. Many kinds of labelings have been studied and an excellent survey of graph labeling can be found in [4].

Sedlacek [13] introduced the magic labeling. A graph $G$ with $q$ edges is magic if the edges of $G$ can be labeled by the numbers $1, 2, 3, \ldots, q$ so that the sum of labels of all the edges incident with any vertex is the same. MacDougall et al. [10] introduced the notion of a vertex magic total labeling. If $G$ is a finite simple undirected graph with $p$ vertices and $q$ edges, then a vertex
magic total labeling is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, p + q\}$ with the property that for every $u$ in $V(G)$, $f(u) + \sum_{v \in N(u)} f(uv) = k$, for some constant $k$. They studied the basic properties of vertex magic graphs and showed some families of graphs having vertex magic total labeling.

MacDougall et al. [9] further introduced the super vertex magic total labeling. They called a vertex magic total labeling is super if $f(V(G)) = \{1, 2, 3, \ldots, p\}$. Swaminathan and Jeyanthi [14] introduced a concept with the name super vertex magic labeling, but with different notion. They call a vertex magic total labeling is super if $f(E(G)) = \{1, 2, 3, \ldots, q\}$. To avoid confusion, Marimuthu and Balakrishnan [11] called a vertex magic total labeling as an $E$-super vertex magic total labeling if $f(E(G)) = \{1, 2, 3, \ldots, q\}$. They studied some basic properties of such a labeling.

An antimagic labeling of a graph with $p$ vertices $q$ edges is a bijection $f : E(G) \to \{1, 2, \ldots, q\}$ such that the values at the vertices are distinct, where the value of $v$ is the sum of the labels on edges incident to $v$. In [5], Hartsfield and Ringel made a conjecture on vertex-antimagic labeling and Martin Baca proposed a conjecture about edge-antimagic vertex labeling [1].

In [3], Bondy and Murty discussed about the importance of digraphs. Although many problems lend themselves naturally to a graph-theoretic formulation, the concept of a graph is sometimes not quite adequate. When dealing with problems of traffic flow, for example, it is necessary to know which roads in the network are one-way, and in which direction traffic is permitted. Clearly, a graph of the network is not of much use in such a situation. What we need is a graph in which each link has an assigned orientation—a directed graph.

In 2008, Bloom et al. [2] extended the idea of magic labeling to directed graphs. A magic labeling of a digraph $D$ is a one-to-one map $f$ from $V(D) \cup A(D)$ onto the set of integers $\{1, 2, \ldots, p + q\}$ in which all the sums

$$f(x) + \sum_{(x,y) \in A} f((x,y))$$

and all the sums

$$f(x) + \sum_{(z,x) \in A} f((z,x))$$

are constant, independent of the choice of $x$. A digraph with a magic labeling is a magic digraph.

Hefetz et al. [6] defined an antimagic labeling of a directed graph $D$ with $p$ vertices and $q$ arcs as a bijection from the set of arcs of $D$ to the integers $1, 2, \ldots, q$ such that all $p$ oriented vertex sums are pair-wise distinct, where an oriented vertex sum is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. A digraph is called antimagic if it admits an antimagic labeling. In particular, they proved that every digraph, whose underlying undirected graph is “dense” is antimagic, and that almost
every undirected \( d \)-regular graph admits an orientation which is antimagic.

In this paper, we introduce a variation of antimagic labeling of digraphs. We define vertex in-antimagic total labeling of a digraph \( D \) as a bijection \( f : V(D) \cup A(D) \to \{1,2,3,\ldots,p+q\} \) with the property that

\[
\left\{ w(v) = f(v) + \sum_{u \in I(v)} f((u,v)) : v \in V(D) \right\}
\]

consists of distinct integers. Such a labeling is called a \( V \)-super vertex in-antimagic total digraph (\( V \)-SVIAMT labeling) if \( f(V(D)) = \{1,2,3,\ldots,p\} \). A digraph \( D \) is called a \( V \)-super vertex in-antimagic total digraph (\( V \)-SVIAMT digraph) if \( D \) admits a \( V \)-SVIAMT labeling. The digraph given in Figure 1 is \( V \)-SVIAMT digraph.

\[\text{Figure 1: A } V \text{-SVIAMT digraph}\]

In this paper, we establish such labelings for some digraphs. In particular, we discuss the \( V \)-SVIAMT labelings of generalized de Bruijn digraphs. The notion of generalized de Bruijn digraphs was introduced by Imase and Itoh [7, 8]. These digraphs have been widely studied as topologies for interconnection networks.

The generalized de Bruijn digraph \( G_B(n,d) \) is defined as follows.

Let \( n \) and \( d \) be positive integers with \( d \geq 2 \) and \( n \geq d \). Then

\[
V(G_B(n,d)) = \{0,1,2,\ldots,n-1\} \quad \text{and} \quad A(G_B(n,d)) = \{(x,y)|y \equiv dx + i(\mod n), \quad 0 \leq i \leq d-1\}.
\]

The digraphs \( G_B(2,2), G_B(3,3), G_B(4,2) \) and \( G_B(6,3) \) are given in Figure 2.
2 V-SVIAMT Digraphs

This section deals with V-SVIAMT labeling for some families of digraphs.

Definition 2.1. The unidirectional path \( \vec{P}_p \) is defined as a digraph with

\[
V(\vec{P}_p) = \{v_1, v_2, \ldots, v_p\} \quad \text{and} \quad A(\vec{P}_p) = \{a_i = v_iv_{i+1} | 1 \leq i \leq p - 1\}.
\]

Definition 2.2. The bidirectional path \( \vec{P}_p \) is defined as a digraph with

\[
V(\vec{P}_p) = \{v_1, v_2, \ldots, v_p\} \quad \text{and} \quad A(\vec{P}_p) = \{a_i = v_iv_{i+1} | 1 \leq i \leq p - 1\} \cup \{a_j = v_{j+1}v_j | 1 \leq j \leq p - 1\}.
\]

Definition 2.3. The unidirectional cycle \( \vec{C}_p \) is defined as digraph with

\[
V(\vec{C}_p) = \{v_1, v_2, \ldots, v_p\} \quad \text{and} \quad A(\vec{C}_p) = \{a_i = v_iv_{i+1} | 1 \leq i \leq p\},
\]

where the subscript is taken modulo \( p \).

Definition 2.4. The bidirectional cycle \( \vec{C}_p \) is defined as a digraph with \( V(\vec{C}_p) = \{v_1, v_2, \ldots, v_p\} \) and \( A(\vec{C}_p) = \{a_i = v_iv_{i+1} | 1 \leq i \leq p\} \cup \{a_j = v_{j+1}v_j | 1 \leq j \leq p\} \), where the subscript is taken modulo \( p \).

Theorem 2.5. The unidirectional path \( \vec{P}_p \) is V-SVIAMT.

Proof. Let \( V(\vec{P}_p) \cup A(\vec{P}_p) \to \{1, 2, \ldots, p + q\} \) by \( f(v_i) = i, 1 \leq i \leq p \) and \( f(v_iv_{i+1}) = p + i, 1 \leq i \leq p - 1 \). Clearly, \( f(V(\vec{P}_p)) = \{1, 2, 3, \ldots, p\} \) and
\(f(\overrightarrow{A(P_p)}) = \{p + 1, p + 2, p + 3, \ldots, 2p - 1 = p + q\}\). Now for \(i = 1\), \(w(v_1) = f(v_1) = 1\).

For \(2 \leq i \leq p\),

\[
\begin{align*}
w(v_i) &= f(v_i) + f(v_{i-1}v_i) \\
&= (i) + (p + i - 1) \\
&= p - 1 + 2i.
\end{align*}
\]

Clearly, the set \(\{w(v_i)|2 \leq i \leq p\}\) consists of \(p - 1\) distinct integers.

Suppose \(w(v_i) = w(v_j)\) for some \(i, 2 \leq i \leq p\). Then \(p + 2i = 2\), so that \(2 \geq p + 4\), since \(i \geq 2\). That is, \(p \leq -2\), a contradiction. Therefore, \(\{w(v_i)|1 \leq i \leq p\}\) consists of \(p\) distinct integers.

Thus \(f\) is a V-SVIAMT labeling of \(\overrightarrow{P_p}\), which shows that \(\overrightarrow{P_p}\) is V-SVIAMT.

\(\Box\)

**Theorem 2.6.** The bidirectional path \(\overrightarrow{P_p}\) is V-SVIAMT.

**Proof.** Let \(V(\overrightarrow{P_p}) = \{v_1, v_2, \ldots, v_p\}\) and let \(A(\overrightarrow{P_p}) = \{a_i = v_iv_{i+1}|1 \leq i \leq p - 1\}\). Then \(q = 2p - 2\).

Define \(f : V(\overrightarrow{P_p}) \cup A(\overrightarrow{P_p}) \rightarrow \{1, 2, \ldots, p + q\}\) by

\[
f(v_1) = p; \quad f(v_i) = i - 1, 1 \leq i \leq p; \quad f(v_i + 1) = p - 1 + i, 1 \leq i \leq p - 1.
\]

Clearly, \(f(\overrightarrow{V(P_p)}) = \{1, 2, 3, \ldots, p\}\) and \(f(\overrightarrow{A(P_p)}) = \{p + 1, p + 2, p + 3, \ldots, 2p - 1, 2p, 2p + 1, \ldots, 3p - 2 = p + q\}\). Now for \(i = 1\), \(w(v_1) = f(v_1) + f(v_2v_1) = p + 2p = 3p\). For \(i = p\), \(w(v_p) = f(v_p) + f(v_{p-1}v_p) = (p - 1) + (2p - 1) = 3p - 2\).

For \(2 \leq i \leq p - 1\),

\[
\begin{align*}
w(v_i) &= f(v_i) + f(v_{i-1}v_i) + f(v_{i+1}v_i) \\
&= (i - 1) + (p + i - 1) + (2p - 1 + i) \\
&= 3p - 3 + 3i.
\end{align*}
\]

Clearly, \(w(v_1) \neq w(v_p)\) and the set \(\{w(v_i)|2 \leq i \leq p - 1\}\) consists of \(p - 2\) distinct integers.

Suppose \(w(v_i) = w(v_j)\) for some \(i, 2 \leq i \leq p - 1\). Then \(3p - 3 + 3i = 3p\), so that \(i = 1\), a contradiction. Thus \(w(v_i) \neq w(v_1)\), for any \(i, 2 \leq i \leq p - 1\).

Suppose \(w(v_i) = w(v_p)\), for some \(i, 2 \leq i \leq p - 1\). Then \(3p - 3 + 3i = 3p - 2\), so that \(i = \frac{1}{3}\), a contradiction. Therefore, \(w(v_i) \neq w(v_p)\), for any \(i, 2 \leq i \leq p - 1\). Thus \(\{w(v_i)|1 \leq i \leq p\}\) consists of \(p\) distinct integers, which shows that \(f\) is a V-SVIAMT labeling of \(\overrightarrow{P_p}\). 

\(\Box\)
Theorem 2.7. The unidirectional cycle \( \overrightarrow{C_p} \) is V-SVIAMT.

Proof. Let \( V(\overrightarrow{C_p}) = \{v_1, v_2, \ldots, v_p\} \) and let \( A(\overrightarrow{C_p}) = \{a_i = v_iv_{i+1}|1 \leq i \leq p\} \), where the subscript is taken modulo \( p \). Then \( q = p \).

Define \( f : V(\overrightarrow{C_p}) \cup A(\overrightarrow{C_p}) \rightarrow \{1, 2, \ldots, p + q\} \) by

\[
\begin{align*}
    f(v_i) &= i, 1 \leq i \leq p \\
    f(v_{i+1}) &= p + i + 1, 1 \leq i \leq p - 1; \\
    f(v_pv_1) &= p + 1.
\end{align*}
\]

Clearly, \( f(V(\overrightarrow{C_p})) = \{1, 2, 3, \ldots, p\} \) and \( f(A(\overrightarrow{C_p})) = \{p + 1, p + 2, p + 3, \ldots, 2p = p + q\} \). Now for \( i = 1 \), \( w(v_1) = f(v_1) + f(v_pv_1) = 1 + p + 1 = p + 2 \).

For \( 2 \leq i \leq p \),

\[
\begin{align*}
    w(v_i) &= f(v_i) + f(v_{i-1}v_i) \\
    &= (i) + (p + i) \\
    &= p + 2i.
\end{align*}
\]

Clearly, the set \( \{w(v_i)|2 \leq i \leq p\} \) consists of \( p - 1 \) distinct integers.

Suppose \( w(v_i) = w(v_1) \), for some \( i, 2 \leq i \leq p \). Then \( p + 2i = p + 2 \), so that \( i = 1 \), a contradiction. Therefore, \( w(v_i) \neq w(v_1) \), for any \( i, 2 \leq i \leq p \). Thus \( \{w(v_i)|1 \leq i \leq p\} \) consists of \( p \) distinct integers proving the existence of a V-SVIAMT labeling of \( \overrightarrow{C_p} \).

\( \square \)

Theorem 2.8. The bidirectional cycle \( \leftrightarrow C_n \) is V-SVIAMT.

Proof. Let \( V(\leftrightarrow C_n) = \{v_1, v_2, \ldots, v_n\} \). Since every vertex of \( \leftrightarrow C_n \) has in-degree two, let \( a_1^1, a_1^2 \) be the arcs with head \( v_i, 1 \leq i \leq n \).

Clearly \( A(\leftrightarrow C_n) = A_1 \cup A_2 \) where \( A_1 = \{a_1^1|i = 1, 2, \ldots, n\} \) and \( A_2 = \{a_1^2|i = 1, 2, \ldots, n\} \).

The labeling of vertices and arcs is given in Table 1.

<table>
<thead>
<tr>
<th>Arc sets</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( v_1 )</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( n + 1 )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( 3n )</td>
</tr>
</tbody>
</table>

Table 1

From Table 1, the sums of the labels in \( A_1 \) and \( A_2 \) are \( 4n + 1 = k \), a constant for all the vertices. Thus \( \{w(v_i)|1 \leq i \leq n\} = \{k + 1, k + 2, \ldots, k + n\} \) consists of \( n \) consecutive integers. Hence \( \leftrightarrow C_n \) is V-SVIAMT. \( \square \)
3 V-SVIAMT Labelings of Generalized De Bruijn Digraphs

In this section, we investigate the existence of V-SVIAMT labelings for generalized de Bruijn digraphs.

**Theorem 3.1.** The generalized de Bruijn digraphs $G_B(n,d)$ is V-SVIAMT for all $n$ and $d$.

**Proof.** Let $V(G_B(n,d)) = \{v_1, v_2, \ldots, v_n\}$. Since every vertex of $G_B(n,d)$ has in-degree $d$, let $a_1^1, a_1^2, \ldots, a_1^d$ be the arcs with head $v_i, 1 \leq i \leq n$. Then $A(G_B(n,d)) = \bigcup_{j=1}^{d} A_j$ where $A_j = \{a_j^i | i = 1, 2, 3, \ldots, n\}, 1 \leq j \leq d$.

**Case 1.** $d$ is even.

The labelings of the vertices and arcs is given in Table 2.

<table>
<thead>
<tr>
<th>Arc sets</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$\cdots$</th>
<th>$v_{n-1}$</th>
<th>$v_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$n + 1$</td>
<td>$n + 2$</td>
<td>$n + 3$</td>
<td>$\cdots$</td>
<td>$2n - 1$</td>
<td>$2n$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$3n$</td>
<td>$3n - 1$</td>
<td>$3n - 2$</td>
<td>$\cdots$</td>
<td>$2n + 2$</td>
<td>$2n + 1$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$3n + 1$</td>
<td>$3n + 2$</td>
<td>$3n + 3$</td>
<td>$\cdots$</td>
<td>$4n - 1$</td>
<td>$4n$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$5n$</td>
<td>$5n - 1$</td>
<td>$5n - 2$</td>
<td>$\cdots$</td>
<td>$4n + 2$</td>
<td>$4n + 1$</td>
</tr>
<tr>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$A_{d-1}$</td>
<td>$\cdot (d - 1)n + 1$</td>
<td>$\cdot (d - 1)n + 2$</td>
<td>$(d - 1)n + 3$</td>
<td>$\cdots$</td>
<td>$dn - 1$</td>
<td>$dn$</td>
</tr>
<tr>
<td>$A_d$</td>
<td>$(d + 1)n$</td>
<td>$(d + 1)n - 1$</td>
<td>$(d + 1)n - 2$</td>
<td>$\cdots$</td>
<td>$dn + 2$</td>
<td>$dn + 1$</td>
</tr>
</tbody>
</table>

*Table 2*

From the Table 2, the sum of the labels in the first two rows $A_1$ and $A_2$ at each vertex is a constant $4n + 1$. The sum of the labels in the next two rows $A_3$ and $A_4$ at each vertex is a constant $8n + 1$. The sum of the labels in the next two rows $A_5$ and $A_6$ at each vertex is also a constant $12n + 1$ and so on. Therefore the sums of the labels in all the rows at each vertex is a constant, say $k$. Thus $\{w(v_j) | 1 \leq i \leq n\} = \{k + 1, k + 2, \ldots, k + n\}$ consists of $n$ consecutive integers. Hence $G_B(n,d)$ is V-SVIAMT for all $n$, when $d$ is even.

**Case 2.** $d$ is odd.

The labeling of the vertices and arcs is given in Table 3.
Table 3

<table>
<thead>
<tr>
<th>Arc sets</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$v_2$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$n+1$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$3n$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$3n+1$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$5n$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$5n+1$</td>
</tr>
<tr>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$A_{d-1}$</td>
<td>$dn$</td>
</tr>
<tr>
<td>$A_d$</td>
<td>$dn+1$</td>
</tr>
</tbody>
</table>

From the Table 3, the sum of the labels in the two rows $A_2$ and $A_3$ at each vertex is a constant $6n+1$. The sum of the labels in the next two rows $A_4$ and $A_5$ at each vertex is a constant $10n+1$. The sum of the labels in the next two rows $A_3$ and $A_6$ at each vertex is a constant $14n+1$ and so on. Therefore the sums of the labels in all the rows except the first row at each vertex is a constant, say $k$.

The set of the sums of the labels in the first row and the corresponding vertex is $\{n+2, n+4, \ldots, 3n-2, 3n\}$. Thus $\{w(v_i)|1 \leq i \leq n\} = \{k+n+2, k+n+4, \ldots, k+3n-2, k+3n\}$ consists of $n$ distinct integers.

Hence $G_B(n,d)$ is $V$-SVIAMT for all $n$, when $d$ is odd.

4 Conclusion and Scope

In this paper, we have found $V$-SVIAMT labelings of different families of digraphs, particularly generalized de Bruijn digraphs. The definition of $V$-SVIAMT labelings can be modified by using out-neighbors. In this case, we call vertex in-antimagic as vertex out-antimagic (VOAM) and in the first case vertex antimagic as vertex in-antimagic (VIAM). Based on Theorems 2.5 through Theorem 2.8 and Theorem 3.1, we suspect that every digraph may admit a $V$-SVIAMT labeling. It seems to be difficult to solve the following conjecture.

**Conjecture 4.1.** Every digraph is a V-SVIAMT digraph.

References


