SOME RESULTS ON k-EDGE-MAGIC BROKEN WHEEL GRAPHS

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ARTICLE INFO

J. Graph Labeling 1(2) (2015), 65-79.

Article History:
AMS MSC: 05C78
Received : 25.06.2015
Received in revised form : 12.08.2015
Accepted : 20.08.2015
Available online : 30.10.2015

Keywords: Edge-magic, wheel graph, gear graph.
Abstract
Let $G$ be a $(p, q)$-graph in which the edges are labeled $k, k+1, \ldots, k+q-1$, where $k \in \mathbb{Z}$. The vertex sum for a vertex $v$ is the sum of the labels of the incident edges at $v$. If the vertex sums are constant modulo $p$, then $G$ is said to be $k$-edge-magic. In this paper, we give necessary conditions for a family of regular broken wheel graphs to admit $k$-edge-magic labelings. Consequently, we show that some of these conditions are also sufficient.

1 Introduction
All undefined symbols and concepts may be looked up from [1]. Let $G = (V, E)$ be a $(p, q)$-graph, i.e., $|V| = p$ and $|E| = q$. Let $f : E \rightarrow \{k, k+1, \ldots, k+q-1\}$ be a bijection for some $k \in \mathbb{Z}$. The induced mapping $f^+ : V \rightarrow \mathbb{Z}_r$ of $f$ is defined by $f^+(u) = \sum_{uv \in E} f(uv)$ for $u \in V$, the sum is taken in $\mathbb{Z}_r$ for some $r \geq 0$. Note that we fix $\mathbb{Z}_r = \{0, 1, \ldots, r-1\}$ for $r \geq 1$ and denote $\mathbb{Z}$ by $\mathbb{Z}_0$. If $f^+$ is a constant mapping, then $G$ is called $k$-edge-magic over $\mathbb{Z}_r$. If $k = 1$, then $G$ is simply called edge-magic over $\mathbb{Z}_r$, $f$ an edge-magic labeling of $G$ over $\mathbb{Z}_r$ and the value of $f^+$ an edge-magic value of $G$ over $\mathbb{Z}_r$. This concept was first introduced by Shiu and Lee [18] in 2002. Moreover, $G$ being edge-magic over $\mathbb{Z}_p$ or $\mathbb{Z}$ is called edge-magic or supermagic, the labeling $f$ is called an edge-magic labeling or supermagic labeling, respectively. These concepts were introduced by Lee, Seah and Tan [9] in 1992 and Stewart [22] in 1966, respectively. Note that edge-magic value is not unique in general.

The necessary condition (see [16]) for $(p, q)$-graph being $k$-edge-magic is
\[ q(q + 2k - 1) \equiv 0 \pmod{p}. \tag{1.1} \]
Some edge-magic or supermagic graphs were found [5, 9–11, 14–23]. More about supermagic graphs can be found in [2, 4, 6, 7]. For regular graph, there is no different between $k$-edge-magic and edge-magic (see [16, 18, 19]).

Let $[a, b]$ denote the set of integers from $a$ to $b$. Let $S$ and $T$ be multisets of integers. $S \equiv T \pmod{r}$ means that two sets are equal after their elements are taken modulo $r$, where $r \geq 2$. From now on, the term “set” means multiset. Set operations are viewed as multiset operations.

A wheel graph (or wheel, for short) $W_p = C_{p-1} \vee K_1$ is the join graph of the cycle $C_{p-1}$ and the complete graph $K_1$, where $p \geq 4$. So $W_p$ is a $(p, 2p-2)$-graph and hence the
necessary condition for $W_p$-graph being $k$-edge-magic is

$$4k \equiv 6 \pmod{p}. \quad (1.2)$$

Fukuchi [3] showed that wheel graph with $p$ vertices is edge-magic if integer $p \geq 5$, and $p \not\equiv 0 \pmod{4}$. However, the term ‘edge-magic’ in [3] is different from edge-magic in this paper. Lee, Wong and Lo [12] studied on the $Q(a)$-balance edge-magic graphs and provided some results, such as $W_5$, $W_7$, $W_8$ and $W_9$ are strong balance edge-magic. They also showed that all wheels are not edge-magic since (1.1) is not satisfied. By using (1.2) it is easy to obtain:

**Proposition 1.1** ([13]). If the wheel graph $W_p$ is $k$-edge-magic, then we have the following three cases:

1. $p = 4h + 1$ and $k \equiv 2h + 2 \pmod{p}$;
2. $p = 4h + 3$ and $k \equiv 2h + 3 \pmod{p}$;
3. $p = 4h + 2$ and $k \equiv h + 2$ or $3h + 3 \pmod{p}$.

Lee, Su and Wang [13] showed the converse for cases 1 and 2. They also showed that $W_6$ is $k$-edge-magic for all $k \equiv 0, 3 \pmod{6}$, as an example. They conjectured that the converse of case 3 holds. It is still open.

It is easy to obtain the following proposition.

**Proposition 1.2.** Suppose $G$ is a graph of order $p$ and $k \in \mathbb{Z}$. Then $G$ is $k$-edge-magic if and only if $G$ is $(k + pt)$-edge-magic for $t \in \mathbb{Z}$.

So from now on we always assume that $1 \leq k \leq p$.

## 2 Broken wheel graphs

In this paper, we shall study the $k$-edge-magicness of some broken wheel graphs. Let us introduce some definitions and notation first.

Let $V(W_p) = \{c, u_1, \ldots, u_{p-1}\}$, where $u_1u_2 \cdots u_{p-1}u_1$ is a $(p-1)$-cycle and $c$ is the hub (or the center) of the wheel, i.e., $\deg(c) = p - 1 = \Delta(W_p)$. The edge $cu_i$, $1 \leq i \leq p-1$ is called a spoke of the wheel. Let $S = \{cu_i \mid 1 \leq i \leq p-1\}$ be the set of all spokes. Let $\varnothing \neq A \subset S$. The graph $W_p(A) = W_p - (S \setminus A)$ is called a broken wheel graph (or broken wheel, for short). We shall keep these notation throughout this paper.

Let $w$ be a factor of $p - 1$ with $w \geq 2$. Let $A_w = \{cu_{iw+1} \mid 0 \leq i \leq (p - 1)/w - 1\}$. The graph $RW_p(w) = W_p(A_w)$ is called a regular broken wheel. Note that $RW_p(w)$ is a
Some results on \( k \)-edge-magic broken wheel graphs

\((p, q)\)-graph, where \( q = (w + 1)(p - 1)/w \). Clearly \( p \leq q < 2p \). The following figures show \( W_9 \), \( RW_9(2) \), \( RW_9(4) \) and \( RW_9(8) \).

![Figure 1](image)

By pigeonhole principle, it is easy to show that

**Lemma 2.1.** Let \( p \geq 4 \), \( w \mid (p - 1) \) with \( w \geq 2 \) and \( q = (w + 1)(p - 1)/w \). Let \( L = [k, k + q - 1] \) for some \( k \). Then every element in \( L \) is congruent modulo \( p \) to at most one other element in \( L \). Moreover, \( L = \mathbb{Z}_p \cup R \), where \( R \subset \mathbb{Z}_p \).

**Theorem 2.2.** If \( RW_p(w) \) is \( k \)-edge-magic, then \( w = 2 \).

**Proof.** Suppose \( w \geq 5 \) and \( f \) is a \( k \)-edge-magic labeling of \( RW_p(w) \). Since \( f^+(u_2) = f^+(u_3) = f^+(u_4) = f^+(u_5) \) in \( \mathbb{Z}_p \), \( f(u_1u_2) = f(u_3u_4) = f(u_5u_6) \) in \( \mathbb{Z}_p \). But since the image of \( f \) is \( L = [k, k + q - 1] \), where \( q = (w + 1)(p - 1)/w \), by Lemma 2.1 we get a contradiction. So \( 2 \leq w \leq 4 \).

Suppose \( w = 4 \). Consider the subpath \( u_{1+4i}u_{2+4i}u_{3+4i}u_{4+4i}u_{5+4i} \) of the cycle \( C = u_1u_2 \cdots u_{p-1}u_1, \ 0 \leq i \leq (p - 5)/4 \) (here \( u_p = u_1 \)). By a similar argument above we obtain that \( f(u_{1+4i}u_{2+4i}) = f(u_{3+4i}u_{4+4i}) \) and \( f(u_{2+4i}u_{3+4i}) = f(u_{4+4i}u_{5+4i}) \) in \( \mathbb{Z}_p \). Since \( f \) is a bijection, by Lemma 2.1 \( L = Q \cup Q \mod p \) for some \( Q \subset \mathbb{Z}_p \). But it is impossible by Lemma 2.1 again.

Suppose \( w = 3 \). Consider the subpath \( u_{1+3i}u_{2+3i}u_{3+3i}u_{4+3i} \) of the cycle \( C, \ 0 \leq i \leq (p - 4)/3 \). Similarly, we have \( f(u_{1+3i}u_{2+3i}) = f(u_{3+3i}u_{4+3i}) \) in \( \mathbb{Z}_p \). Then

\[
\{ f(u_{1+3i}u_{2+3i}), f(u_{3+3i}u_{4+3i}) \mid 0 \leq i \leq (p - 4)/3 \} \equiv Q \cup Q \mod p,
\]

where \( Q \subset \mathbb{Z}_p \) with \( |Q| = (p - 1)/3 \). By viewing \( Q \) as a subset of \( \mathbb{Z}_2 \), let \( r \) be the largest integer in \( Q \). Then there is an edge \( e \) such that \( f(e) = r + p \). Since \( |Q| = (p - 1)/3, \ r + p \geq (k + (p - 1)/3 - 1) + p = k + q \) which is impossible.

By the above theorem, we only focus on \( w = 2 \). Thus if \( RW_p(2) \) is \( k \)-edge-magic, then \( p = 2n + 1 \) for some \( n \geq 2 \). In some articles, for example \([8]\), \( RW_{2n+1}(2) \) is also called a gear graph and denoted by \( G_n \). For simplicity, we shall use this notation for the rest of this paper.
3 Property of edge-magic gear graphs

Suppose \( f : E(G_n) \to L = [k, k + 3n - 1] \) is a \( k \)-edge-magic labeling of \( G_n \). We let \( L = \mathbb{Z}_{2n+1} \cup R \pmod{2n+1} \) for some \( R \subset \mathbb{Z}_{2n+1} \). Note that \( |R| = n - 1 \). For convenience, we let \( u_0 = u_{2n} \) and \( u_{2n+1} = u_1 \). We shall keep these notation throughout this paper.

**Theorem 3.1.** For \( n \geq 2 \), if \( f \) is a \( k \)-edge-magic labeling of \( G_n \), then \( f^+ = 0 \).

**Proof.** Suppose \( f^+ = s \) for some \( s \in \mathbb{Z}_{2n+1} \). Then we have

\[
\begin{align*}
f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) & \equiv s \pmod{2n+1}, \quad 1 \leq i \leq n; \\
f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(cu_{2i-1}) & \equiv s \pmod{2n+1}, \quad 1 \leq i \leq n; \\
\sum_{i=1}^{n} f(cu_{2i-1}) & \equiv s \pmod{2n+1}.
\end{align*}
\]

Subtracting (3.2) by (3.1) we have

\[
f(u_{2i-2}u_{2i-1}) + f(cu_{2i-1}) - f(u_{2i}u_{2i+1}) \equiv 0 \pmod{2n+1}.
\]

Hence

\[
\sum_{i=1}^{n} f(u_{2i-2}u_{2i-1}) + \sum_{i=1}^{n} f(cu_{2i-1}) - \sum_{i=1}^{n} f(u_{2i}u_{2i+1}) \equiv 0 \pmod{2n+1}.
\]

So we have \( \sum_{i=1}^{n} f(cu_{2i-1}) \equiv 0 \pmod{2n+1} \).

Hence from (3.3) we have \( s \equiv 0 \pmod{2n+1} \). \( \square \)

Following we show the necessary condition for \( G_{(p-1)/2} = RW_p(2) \) being \( k \)-edge-magic.

**Lemma 3.2.** Suppose \( G_n \) is \( k \)-edge-magic for \( n \geq 2 \), where \( 1 \leq k \leq 2n + 1 \). We have

1. if \( n = 6h \), then \( k = 9h + 2 \);
2. if \( n = 6h + 1 \), then \( k = 3h + 2, 7h + 3 \) or \( 11h + 4 \);
3. if \( n = 6h + 2 \), then \( k = 9h + 5 \);
4. if \( n = 6h + 3 \), then \( k = 3h + 3 \);
5. if \( n = 6h + 4 \), then \( k = h + 2, 5h + 5 \) or \( 9h + 8 \);
6. if \( n = 6h + 5 \), then \( k = 3h + 4 \).
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**Proof.** From (1.1) we have
\[
\frac{3}{2}(p - 1) \left( \frac{3}{2}(p - 1) + 2k - 1 \right) \equiv 0 \pmod{p}.
\] (3.4)

1. When \( n = 6h \), from (3.4), we have
\[
(18h)(18h + 2k - 1) \equiv 0 \pmod{12h + 1}.
\]
Equivalently,
\[
(6h - 1)(6h + 2k - 2) \equiv 0 \pmod{12h + 1}.
\]
This implies that
\[
(-3)(3h + k - 1) \equiv 0 \pmod{12h + 1}.
\]
Thus
\[
k \equiv 9h + 2 \pmod{12h + 1}.
\]

2. When \( n = 6h + 1 \), from (3.4), we have
\[
(18h + 3)(18h + 2k + 2) \equiv 0 \pmod{12h + 3}
\]
which implies that
\[
3k \equiv -3h + 3 \pmod{12h + 3}.
\]
Thus
\[
k \equiv -h + 1 \equiv 3h + 2 \pmod{4h + 1}.
\]

So \( k = 3h + 2, 7h + 3 \) or \( 11h + 4 \).

3. The proofs of the remaining cases are similar. We leave to readers.

\[\square\]

**Example 3.1.** Following are two \( k \)-edge-magic labelings of the graphs \( G_3 \) and \( G_5 \) for a suitable \( k \), respectively.

3-edge-magic labeling for \( G_3 \). 4-edge-magic labeling for \( G_5 \).

![Figure 2](image)

For convenience, we will represent the labeling \( f \) of the gear graph \( G_n \) by the following list:

<table>
<thead>
<tr>
<th>( f(u_{2n}u_1) )</th>
<th>( f(u_1u_2) )</th>
<th>( f(u_2u_3) )</th>
<th>( f(u_3u_4) )</th>
<th>[ \vdots ]</th>
<th>( f(u_{2n-1}u_{2n}) )</th>
<th>( f(u_{2n}u_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cdot )</td>
<td>( f(cu_1) )</td>
<td>[ \cdot ]</td>
<td>( f(cu_3) )</td>
<td>[ \cdot ]</td>
<td>( f(cu_{2n-1}) )</td>
<td>( \cdot )</td>
</tr>
</tbody>
</table>

So for the 4-edge-magic labeling of \( G_5 \) above, we represent it as

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>6</th>
<th>17</th>
<th>16</th>
<th>8</th>
<th>14</th>
<th>18</th>
<th>15</th>
<th>7</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \cdot ]</td>
<td>13</td>
<td>[ \cdot ]</td>
<td>10</td>
<td>[ \cdot ]</td>
<td>9</td>
<td>[ \cdot ]</td>
<td>12</td>
<td>[ \cdot ]</td>
<td>11</td>
<td>[ \cdot ]</td>
</tr>
</tbody>
</table>
After taking modulo 11, the set of edge labels \( L = [4, 18] \equiv \mathbb{Z}_{11} \cup [4, 7] \) (mod 11). So we will represent the list above as

\[
\begin{array}{cccccccc}
4 & 5 & 6 & 8 & 3 & 7 & 4 & 4 \\
2 & 10 & 9 & 1 & 0 & & & \\
\end{array}
\]

**Lemma 3.3.** Suppose \( f : E(G_n) \to \mathbb{Z}_{2n+1} \cup R \) is a \( k \)-edge-magic labeling. Suppose \( 0 \notin R \). Then \( f(cu_{2i+1}) = 0 \) for some \( i \).

*Proof.* Suppose an edge \( e \) is incident with a vertex \( v \) of degree 2 and \( f(v) = 0 \). Since \( f^+ = 0 \), another edge incident with \( v \) must be labeled by 0. Thus, \( f(cu_{2i+1}) = 0 \) for some \( i \).

**Lemma 3.4.** Suppose \( f : E(G_n) \to \mathbb{Z}_{2n+1} \cup R \) is a \( k \)-edge-magic labeling. Suppose \( 0 \in R \). Then either \( f(cu_{2i+1}) = 0 = f(cu_{2j+1}) \) for some distinct \( i, j \) or \( f(u_{2i-1}u_{2i}) = f(u_2u_{2i+1}) = 0 \) for some \( i \).

*Proof.* By the proof of Lemma 3.3, if an edge \( e \) in the outer cycle of \( G_n \) is labeled by 0, then the edge with a common vertex of degree 2 with \( e \) is also labeled by 0. This completes the proof since there are exactly two zeros in the list of edge labels.

For some cases, the necessary condition showed in Lemma 3.2 are not sufficient. Let us show you some examples. All arithmetics about the labels are taken in \( \mathbb{Z}_{2n+1} \).

**Example 3.2.** Let \( f \) be a \( k \)-edge-magic labeling of \( G_2 \). According to the notation in Lemma 3.2, \( h = 0 \) and \( k = 5 \). Then \( L \equiv \mathbb{Z}_5 \cup \{0\} \) (mod 5). Up to isomorphic and by Lemma 3.4 we may assume that \( f(cu_1) = f(cu_3) = 0 \). It is easy to see that \( f \) does not exist.

**Example 3.3.** Let \( f \) be a \( k \)-edge-magic labeling of \( G_4 \). According to the notation in Lemma 3.2, \( h = 0 \) and \( k = 2, 5 \) or 8.

Suppose \( k = 2 \). Then \( L \equiv \mathbb{Z}_9 \cup \{2, 3, 4\} \). By Lemma 3.3, without loss of generality we may assume \( f(cu_1) = 0 \). Suppose \( f(u_1u_2) = a \neq 0 \). Then \( f(u_2u_3) \equiv f(u_8u_1) \equiv -a \) and \( f(u_7u_8) \equiv a \). But there is no such \( a \) in \( L \).

By a similar proof as above, we can show that there is no 5-edge-magic labeling of \( G_4 \).

Suppose \( k = 8 \). Then \( L \equiv \mathbb{Z}_9 \cup \{8, 0, 1\} = \mathbb{Z}_9 \cup R \). By Lemma 3.4, without loss of generality, either \( f(cu_1) = 0 = f(cu_{2i-1}) \) with \( i = 2, 3 \) or \( f(u_1u_2) = f(u_2u_3) = 0 \). If \( f(cu_1) = 0 = f(cu_5) \), then \( f(u_1u_2) = f(u_3u_4) = f(u_7u_8) \) which is impossible. If \( f(cu_1) = 0 = f(cu_5) \), then \( f(u_1u_2) = f(u_3u_4) = f(u_8u_1) = f(u_2u_3) = f(u_3u_4) = f(u_5u_6) \) and \( f(u_4u_5) = f(u_6u_7) \). It is impossible since \( |R| = 3 \). Suppose \( f(u_1u_2) = f(u_2u_3) = 0 \).

Let \( f(u_3u_4) = a \), and \( f(u_7u_8) = b \). Then \( f(u_4u_5) = -a, f(u_8u_1) = -b, f(cu_1) = b \) and \( f(cu_3) = -a \). So \( \{-a, b\} \equiv \{1, 8\} \). That means \( a - b = 0 \) or equivalent to \( a = b \). It is impossible by Lemma 2.1. So there is no 8-edge-magic labeling of \( G_4 \).
Theorem 3.5. For \( h \geq 0 \), \( G_{6h+3} \) is \((3h + 3)\)-edge-magic.

Proof. For \( h = 0 \), it is shown in Example 3.1. So we assume that \( h \geq 1 \). Again the arithmetics about the labels are taken in \( \mathbb{Z}_{12h+7} \). So we omit to write \( \mod 12h+7 \).

Case 1. Suppose \( h = 2m \) for \( m \geq 1 \). Then \( L = [6m + 3, 42m + 11] \). An edge-labeling \( f \) is defined by

\[
\begin{align*}
    f(u_{4t+1}u_{4t+2}) &= \begin{cases} 
    9m + 3 + t, & \text{if } 0 \leq t \leq 3m; \\
    39m + 11 - t, & \text{if } 3m + 1 \leq t \leq 6m + 1,
    \end{cases} \quad (3.5) \\
    f(u_{4t+2}u_{4t+3}) &= \begin{cases} 
    15m + 4 - t, & \text{if } 0 \leq t \leq 3m; \\
    33m + 10 + t, & \text{if } 3m + 1 \leq t \leq 6m + 1,
    \end{cases} \quad (3.6) \\
    f(u_{4t+3}u_{4t+4}) &= \begin{cases} 
    9m + 2 - t, & \text{if } 0 \leq t \leq 3m - 1; \\
    27m + 9 + t, & \text{if } 3m \leq t \leq 6m,
    \end{cases} \quad (3.7) \\
    f(u_{4t+4}u_{4t+5}) &= \begin{cases} 
    15m + 5 + t, & \text{if } 0 \leq t \leq 3m; \\
    45m + 12 - t, & \text{if } 3m + 1 \leq t \leq 6m,
    \end{cases} \quad (3.8) \\
    f(cu_i) &= \begin{cases} 
    24m + 7 - 2i, & \text{if } j = 4i + 1, 1 \leq i \leq 3m; \\
    12m + 4 + 2i, & \text{if } j = 4i + 1, 3m + 1 \leq i \leq 6m + 1; \\
    24m + 8 + 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m; \\
    36m + 9 - 2i, & \text{if } j = 4i + 3, 3m + 1 \leq i \leq 6m + 1.
    \end{cases} \quad (3.9)
\end{align*}
\]

We can check that the image of \( f \) is

\[
[9m + 3, 12m + 3] \cup [33m + 10, 36m + 10] \text{ from } (3.5);
\]

\[
[12m + 4, 15m + 4] \cup [36m + 11, 39m + 11] \text{ from } (3.6);
\]

\[
[6m + 3, 9m + 2] \cup [30m + 9, 33m + 9] \text{ from } (3.7);
\]

\[
[15m + 5, 18m + 5] \cup [39m + 12, 42m + 11] \text{ from } (3.8); \text{ and}
\]

\[
[18m + 6, 30m + 8] \text{ from } (3.9).
\]

So \( f \) is a bijection.

Now we are going to check that \( f^+ = 0 \).

\[
f^+(u_{4t+4}) = f(u_{4t+3}u_{4t+4}) + f(u_{4t+4}u_{4t+5})
\]

\[
= \begin{cases} 
    (9m + 2 - t) + (15m + 5 + t) = 24m + 7, & \text{if } 0 \leq t \leq 3m - 1 \\
    (27m + 9 + 3m) + (15m + 5 + 3m) = 48m + 14, & \text{if } t = 3m \\
    (27m + 9 + t) + (45m + 12 - t) = 72m + 21, & \text{if } 3m + 1 \leq t \leq 6m
    \end{cases}
\]

\[
\equiv 0.
\]

Similarly, we can verify that \( f^+(u_{4t+2}) = 0 \) for \( 0 \leq t \leq 6m + 1 \).

\[
f^+(u_1) = f^+(u_{24m+7}) = f(u_{24m+6}u_1) + f(cu_{24m+7}) + f(u_1u_2)
\]

\[
= (33m + 10 + 6m + 1) + (36m + 9 - 12m - 2) + (9m + 3 + 0)
\]

\[
= 72m + 21 \equiv 0.
\]
Case 2. Suppose \( G \) according to the proof above, we have a 9-edge-magic labeling for \( \text{order of any gear graph is odd and from (1.1), the sum of labels is 0. Thus } f^+(e) = 0. \]

Similarly, we can verify that \( f^+(u_{4t+1}) = 0 \) for \( 0 \leq t \leq 6m+1 \). Finally, since the order of any gear graph is odd and from (1.1), the sum of labels is 0. Thus \( f^+(e) = 0. \]

Case 2. Suppose \( h = 2m + 1 \) for \( m \geq 0 \). We only define the labeling \( f \) for this case. To show \( f^+ \) being an edge-magic labeling is similar to Case 1. It is left to readers.

\[
f(u_{4t+1}u_{4t+2}) = \begin{cases} 
9m + 7 - t, & \text{if } 0 \leq t \leq 3m + 1; \\
27m + 22 + t, & \text{if } 3m + 2 \leq t \leq 6m + 4,
\end{cases}
\]

\[
f(u_{4t+2}u_{4t+3}) = \begin{cases} 
15m + 12 + t, & \text{if } 0 \leq t \leq 3m + 2; \\
45m + 35 - t, & \text{if } 3m + 3 \leq t \leq 6m + 4,
\end{cases}
\]

\[
f(u_{4t+3}u_{4t+4}) = \begin{cases} 
9m + 8 + t, & \text{if } 0 \leq t \leq 3m + 1; \\
39m + 30 - t, & \text{if } 3m + 2 \leq t \leq 6m + 3,
\end{cases}
\]

\[
f(u_{4t+4}u_{4t+5}) = \begin{cases} 
15m + 11 - t, & \text{if } 0 \leq t \leq 3m + 1; \\
33m + 27 + t, & \text{if } 3m + 2 \leq t \leq 6m + 3,
\end{cases}
\]

\[
f(cu_i) = \begin{cases} 
24m + 19 + 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 2; \\
36m + 28 - 2i, & \text{if } j = 4i + 1, 3m + 3 \leq i \leq 6m + 4; \\
24m + 18 - 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 1; \\
12m + 11 + 2i, & \text{if } j = 4i + 3, 3m + 2 \leq i \leq 6m + 3.
\end{cases}
\]

This completes the proof.

**Example 3.4.** According to the proof above, we have a 9-edge-magic labeling for \( G_{15} \)

\[
\begin{array}{ccccccccccccccc}
\cdot & 31 & \cdot & 32 & \cdot & 29 & \cdot & 34 & \cdot & 27 & \cdot & 36 & \cdot & 25 & \cdot & 38 \\
\end{array}
\]

\[
\begin{array}{ccccccccccccccc}
23 & 46 & 47 & 40 & 53 & 45 & 48 & 41 & 52 & 44 & 49 & 51 & 43 & 50 \\
\cdot & 24 & \cdot & 37 & \cdot & 26 & \cdot & 35 & \cdot & 28 & \cdot & 33 & \cdot & 30 & \cdot \\
\end{array}
\]

and a 6-edge-magic labeling of \( G_9 \)

\[
\begin{array}{ccccccccccccccc}
\cdot & 19 & \cdot & 18 & \cdot & 21 & \cdot & 16 & \cdot & 23 & \cdot & 15 & \cdot & 22 & \cdot & 17 & \cdot & 20 & \cdot \\
\end{array}
\]
Some results on $k$-edge-magic broken wheel graphs

The proofs of the following theorems are similar to that of Theorem 3.5, so we only provide corresponding labelings and omit the proofs.

**Theorem 3.6.** For $h \geq 0$, $G_{6h+5}$ is $(3h+4)$-edge-magic.

**Proof.** Suppose $h = 2m$ for $m \geq 0$. Define $f$ by

\[
\begin{align*}
    f(u_{4t+1}u_{4t+2}) &= \begin{cases} 
        9m + 4 - t, & \text{if } 0 \leq t \leq 3m; \\
        27m + 13 + t, & \text{if } 3m + 1 \leq t \leq 6m + 2,
    \end{cases} \\
    f(u_{4t+2}u_{4t+3}) &= \begin{cases} 
        15m + 7 + t, & \text{if } 0 \leq t \leq 3m + 1; \\
        45m + 20 - t, & \text{if } 3m + 2 \leq t \leq 6m + 2,
    \end{cases} \\
    f(u_{4t+3}u_{4t+4}) &= \begin{cases} 
        9m + 5 + t, & \text{if } 0 \leq t \leq 3m; \\
        39m + 17 - t, & \text{if } 3m + 1 \leq t \leq 6m + 1,
    \end{cases} \\
    f(u_{4t+4}u_{4t+5}) &= \begin{cases} 
        15m + 6 - t, & \text{if } 0 \leq t \leq 3m; \\
        33m + 16 + t, & \text{if } 3m + 1 \leq t \leq 6m + 1,
    \end{cases} \\
    f(cu_j) &= \begin{cases} 
        24m + 11 + 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 1; \\
        36m + 16 - 2i, & \text{if } j = 4i + 1, 3m + 2 \leq i \leq 6m + 2; \\
        24m + 10 - 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m; \\
        12m + 7 + 2i, & \text{if } j = 4i + 3, 3m + 1 \leq i \leq 6m + 1.
    \end{cases}
\end{align*}
\]
Suppose \( h = 2m + 1 \) for \( m \geq 0 \). Define \( f \) by

\[
\begin{align*}
f(u_{4t+1}u_{4t+2}) &= \begin{cases} 
9m + 9 + t, & \text{if } 0 \leq t \leq 3m + 2; \\
39m + 37 - t, & \text{if } 3m + 3 \leq t \leq 6m + 5,
\end{cases} \\
f(u_{4t+2}u_{4t+3}) &= \begin{cases} 
15m + 14 - t, & \text{if } 0 \leq t \leq 3m + 2; \\
33m + 32 + t, & \text{if } 3m + 3 \leq t \leq 6m + 5,
\end{cases} \\
f(u_{4t+3}u_{4t+4}) &= \begin{cases} 
9m + 8 - t, & \text{if } 0 \leq t \leq 3m + 1; \\
27m + 27 + t, & \text{if } 3m + 2 \leq t \leq 6m + 4,
\end{cases} \\
f(u_{4t+4}u_{4t+5}) &= \begin{cases} 
15m + 15 + t, & \text{if } 0 \leq t \leq 3m + 2; \\
45m + 42 - t, & \text{if } 3m + 3 \leq t \leq 6m + 4,
\end{cases} \\
f(cu_i) &= \begin{cases} 
24m + 23 - 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 2; \\
12m + 12 + 2i, & \text{if } j = 4i + 1, 3m + 3 \leq i \leq 6m + 5; \\
24m + 24 + 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 2; \\
36m + 33 - 2i, & \text{if } j = 4i + 3, 3m + 3 \leq i \leq 6m + 4.
\end{cases}
\]

Example 3.5. According to the construction above, we have a 4-edge-magic labeling for \( G_5 \)

\[
\begin{array}{cccccccccccc}
& 18 & 4 & 7 & 5 & 6 & 14 & 8 & 16 & 17 & 15 & 18 \\
\hline
\end{array}
\]

and a 7-edge-magic labeling for \( G_{11} \)

\[
\begin{array}{cccccccccccccccc}
\hline
\hline
\end{array}
\]

Theorem 3.7. For \( h \geq 1 \), \( G_{6h+1} \) is \((3h + 2)\)-edge-magic.

Proof. Suppose \( h = 2m \) for \( m \geq 1 \). Define \( f \) by

\[
\begin{align*}
f(u_{4t+1}u_{4t+2}) &= \begin{cases} 
9m + 1 - t, & \text{if } 0 \leq t \leq 3m - 1; \\
27m + 4 + t, & \text{if } 3m \leq t \leq 6m,
\end{cases} \\
f(u_{4t+2}u_{4t+3}) &= \begin{cases} 
15m + 2 + t, & \text{if } 0 \leq t \leq 3m; \\
45m + 5 - t, & \text{if } 3m + 1 \leq t \leq 6m,
\end{cases} \\
f(u_{4t+3}u_{4t+4}) &= \begin{cases} 
9m + 2 + t, & \text{if } 0 \leq t \leq 3m - 1; \\
39m + 4 - t, & \text{if } 3m \leq t \leq 6m - 1,
\end{cases} \\
f(u_{4t+4}u_{4t+5}) &= \begin{cases} 
15m + 1 - t, & \text{if } 0 \leq t \leq 3m - 1; \\
33m + 5 + t, & \text{if } 3m \leq t \leq 6m - 1,
\end{cases}
\]

\]
Some results on $k$-edge-magic broken wheel graphs

\[
f(cu_j) = \begin{cases} 
24m + 3 + 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m; \\
36m + 4 - 2i, & \text{if } j = 4i + 1, 3m + 1 \leq i \leq 6m; \\
24m + 2 - 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m - 1; \\
12m + 3 + 2i, & \text{if } j = 4i + 3, 3m \leq i \leq 6m - 1.
\end{cases}
\]

Suppose $h = 2m + 1$ for $m \geq 0$. Define $f$ by

\[
f(u_{4t+1}u_{4t+2}) = \begin{cases} 
9m + 6 + t, & \text{if } 0 \leq t \leq 3m + 1; \\
39m + 24 - t, & \text{if } 3m + 2 \leq t \leq 6m + 3,
\end{cases}
\]

\[
f(u_{4t+2}u_{4t+3}) = \begin{cases} 
15m + 9 - t, & \text{if } 0 \leq t \leq 3m + 1; \\
33m + 21 + t, & \text{if } 3m + 2 \leq t \leq 6m + 3,
\end{cases}
\]

\[
f(u_{4t+3}u_{4t+4}) = \begin{cases} 
9m + 5 - t, & \text{if } 0 \leq t \leq 3m; \\
27m + 18 + t, & \text{if } 3m + 1 \leq t \leq 6m + 2,
\end{cases}
\]

\[
f(u_{4t+4}u_{4t+5}) = \begin{cases} 
15m + 10 + t, & \text{if } 0 \leq t \leq 3m + 1; \\
45m + 27 - t, & \text{if } 3m + 2 \leq t \leq 6m + 2,
\end{cases}
\]

\[
f(cu_j) = \begin{cases} 
24m + 15 - 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 1; \\
12m + 8 + 2i, & \text{if } j = 4i + 1, 3m + 2 \leq i \leq 6m + 3; \\
24m + 16 + 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 1; \\
36m + 21 - 2i, & \text{if } j = 4i + 3, 3m + 2 \leq i \leq 6m + 2.
\end{cases}
\]

Example 3.6. According to the construction above, we have an 8-edge-magic labeling for $G_{13}$

\[
\begin{array}{cccccccccccccccc}
44 & 10 & 17 & 11 & 16 & 9 & 18 & 12 & 15 & 8 & 19 & 13 & 14 & 34
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

and an 8-edge-magic labeling for $G_7$

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
- & 15 & - & 16 & - & 13 & - & 18 & - & 12 & - & 8 & - & 14 & -
\end{array}
\]

We also find some ad hoc examples. We have six 11-edge-magic labelings for $G_6$. They are

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
- & 13 & - & 18 & - & 22 & - & 26 & - & 21 & - & 17 & -
\end{array}
\]
We obtain two 14-edge-magic labelings for $G_8$. They are

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We get one 23-edge-magic labeling for $G_{14}$:

| 60 | 27 | 31 | 34 | 24 | 26 | 32 | 35 | 23 | 25 | 33 | 41 | 46 | 30 | 28 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 29 | 51 | 37 | 49 | 39 | 42 | 40 |    |    |    |    |    |    |    |    |    |

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| 60 | 27 | 31 | 34 | 24 | 26 | 32 | 35 | 23 | 25 | 33 | 41 | 46 | 30 | 28 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 29 | 51 | 37 | 49 | 39 | 42 | 40 |    |    |    |    |    |    |    |    |    |

From those examples, we have not discovered any regulation to obtain edge-magic labelings for other unsolved cases. We summarize those unsolved problems below:

**Problem 3.1.** Find a $(9h + 2)$-edge-magic labeling for $G_{6h}$, $h \geq 1$.

**Problem 3.2.** Find a $(7h + 3)$-edge-magic labeling and a $(11h + 4)$-edge-magic labeling for $G_{6h+1}$, $h \geq 1$.

**Problem 3.3.** Find a $(9h + 5)$-edge-magic labeling for $G_{6h+2}$, $h \geq 1$.

**Problem 3.4.** Find a $(h + 2)$-edge-magic labeling, a $(5h + 5)$-edge-magic labeling, and a $(9h + 8)$-edge-magic labeling for $G_{6h+4}$, $h \geq 1$.

**Acknowledgement**

Authors thank the anonymous referees for their useful comments on the paper which resulted in this nice format.
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References


