PROPERLY EVEN HARMONIOUS LABELINGS OF DISJOINT UNIONS WITH EVEN SEQUENTIAL GRAPHS

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Abstract

A graph G with q edges is said to be harmonious if there exists an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label \( f(x) + f(y) \pmod{q} \), the resulting edge labels are distinct. If G is a tree, exactly one label may be used on two vertices. Over the years, many variations of harmonious labelings have been introduced.

An injective labeling f of a graph G with q edges is even 2a-sequential if the vertex labels are from \( \{0, 1, \ldots, 2q - 1\} \) and the edge labels induced by \( f(u) + f(v) \) for each edge uv are \( 2a, 2a + 2, \ldots, 2a + 2q - 2 \). When G is a tree, the allowable vertex labels are \( 0, 1, \ldots, 2q \).

We study a variant of harmonious labeling. A function f is properly even harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to 2q − 2 and the induced function \( f^* \) from the edges of G to \( 0, 2, \ldots, 2(q - 1) \) defined by \( f^*(xy) = f(x) + f(y) \pmod{2q} \) is bijective.

This paper focuses on finding even harmonious labelings for disjoint graphs where one of these graphs has a 2a-sequential labeling. Among the families we investigate are: the disjoint union of 2a-sequential graphs with paths, the square of paths, caterpillars, and wheels.

1 Introduction

A vertex labeling of a graph G is a mapping f from the vertices of G to a set of elements, often integers. Each edge xy has a label that depends on adjacent vertices x and y and their labels f(x) and f(y). Graph labeling methods began with Rosa [7] in 1967. In 1980, Graham and Sloane [5] introduced harmonious labelings in connection with error-correcting codes and channel assignment problems. There have been three published papers regarding even harmonious graph labelings by Sarasija and Binthiya [8,9] and Gallian and Shoenhard [2]. An extensive survey of graph labeling methods is available online [1]. We follow the notation in [1].

2 Preliminaries

In this section we provide definitions and our motivation for focusing on disconnected graphs.

Definition 2.1. A graph G with q edges is said to be harmonious if there exists an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label \( f(x) + f(y) \pmod{q} \), the resulting
edge labels are distinct. When $G$ is a tree, exactly one edge label may be used on two vertices.

**Definition 2.2.** A function $f$ is said to be an *even harmonious* labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q$ and the induced function $f^*$ from the edges of $G$ to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) \pmod{2q}$ is bijective.

**Definition 2.3.** An even harmonious labeling of a graph $G$ with $q$ edges is said to be a *properly even harmonious labeling* if the vertex labels belong to $\{0, 2, \ldots, 2q - 2\}$.

Because 0 and $2q$ are equal modulo $2q$, Gallian and Schoenhard [2] introduced the following more desirable form of even harmonious labelings.

**Definition 2.4.** A graph that has a (properly) even harmonious labeling is called a (properly) *even harmonious* graph.

**Definition 2.5.** [3] An injective labeling $f$ of a graph $G$ with $q$ edges is *a*-sequential if the vertex labels are from $\{0, 1, \ldots, q - 1\}$ and the edge labels induced by $f(u) + f(v)$ for each edge $uv$ are $a_0, a_1, \ldots, a_q - 1$. When $G$ is a tree, the allowable vertex labels are $0, 1, \ldots, q$.

**Definition 2.6.** An injective labeling $f$ of a graph $G$ with $q$ edges is *even 2a*-sequential if the vertex labels are from $\{0, 1, \ldots, 2q - 1\}$ and the edge labels induced by $f(u) + f(v)$ for each edge $uv$ are $2a, 2a + 2, \ldots, 2a + 2q - 2$. When $G$ is a tree, the allowable vertex labels are $0, 1, \ldots, 2q$.

By taking the edge labels of an a-sequentially labeled graph with $q$ edges modulo $q$, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Numerous families of a-sequential graphs are listed in [1].

For connected a-sequential graphs, a connected 2a-sequential graph can be obtained by multiplying all the vertex labels by 2. Notice that the vertex labels in resulting graph belong to $\{0, 2, \ldots, 2q - 2\}$ (or $\{0, 2, \ldots, 2q\}$ for trees) and the edges labels are from $2a$ to $2a + 2q - 2$. Moreover, a connected a-sequential graph can be obtained from a connected even 2a-sequential graph with even vertex labels by dividing all the vertex labels by 2. Likewise, a 2a-sequential labeling of a connected graph with odd vertex labels induces an a-sequential labeling of the graph by subtracting 1 from each vertex label and dividing by 2. Thus for connected graphs, a-sequential is equivalent to 2a-sequential. Consequently, we focus on even harmonious labelings of disconnected graphs.

### 3 Disconnected Graphs

In this section we give properly even harmonious labelings for the disjoint unions of a 2a-sequential graph and paths, the square of paths, caterpillars, and wheels.
Theorem 3.1. $G \cup P_n^2$ is properly even harmonious if $G$ is an even $2a$-sequential graph and $n > 2$.

Proof. When $G$ has $q$ edges the modulus is $2q + 4n - 6$. Label the vertices of $G$ using the $2a$-sequential labeling. Label the vertices of $P_n^2$ with $a + q - 1, a + q + 1, \ldots, a + q + 2n - 3$ as shown in Figure 1. The corresponding edge labels are the even integers from $2a + 2q$ to $2a + 2q + 4n - 8$.

Since we may assume that the vertex labels of $G$ and $P_n^2$ have opposite parity, their labels have nothing in common. Moreover, because the vertex labels in the $P_n^2$ component are increasing and the gap between the largest vertex label and smallest vertex label of $P_n^2$ is $2n - 2$ and the modulus is $2q + 4n - 6$ there is no wrap around to the first vertex label of $P_n^2$. \qed

Figure 1: $P_7^2$ vertex labeling for Theorem 3.1

Figure 2: $P_9$ vertex labeling for $2a = 8$, $q = 10$, modulo 36, Theorem 3.2, Case 1.a
Theorem 3.1 raises the question that if \( G \) is an even \( 2a \)-sequential graph can we find a properly even harmonious labeling for \( G \cup P_n \) given some conditions on \( n \)?

**Theorem 3.2.** \( G \cup P_n \) is properly even harmonious if \( G \) is an even \( 2a \)-sequential graph when \( n > 1 \) and \( n \equiv 1, 2 \) (mod 4).

**Proof.** When \( G \) has \( q \) edges the modulus is \( 2q + 2n - 2 \). Label the vertices of \( G \) using the \( 2a \)-sequential labeling.

**Step 1:** Arrange \( P_n \) into a bipartite set; denote the set on the left \( L \) and the set on the right \( R \). (See Figure 3.)

- **Case 1:** \( n \equiv 1 \) (mod 4)
  - **Case 1.a:** \( a + q \equiv 0 \) (mod 2).

  **Step 2:** Label the vertices of \( L \) with \( a + q - \frac{n+1}{2}, a + q - \frac{n+1}{2} + 2, \ldots, a + q - \frac{n+1}{2} + n - 1 \).

  **Step 3:** Label the vertices of \( R \) with \( a + q + \frac{n+1}{2}, a + q + \frac{n+1}{2} + 2, \ldots, a + q + \frac{n+1}{2} + n - 3 \).

  The corresponding edge labels are \( 2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4 \) as shown in Figure 2.

  To show no duplication of vertex labels, notice that we need \( a + q - \frac{n+1}{2} + n - 1 < a + q + \frac{n+1}{2} \) which simplifies to \( n - 1 < n + 1 \) which is always true. Hence there are no duplicate vertex labels in the \( P_n \) component.

  - **Case 1.b:** \( a + q \equiv 1 \) (mod 2)

  **Step 2:** Label the vertices of \( L \) with \( a + q - \frac{n+1}{2} - 1, a + q - \frac{n+1}{2} + 1, \ldots, a + q - \frac{n+1}{2} + n - 2 \).

  **Step 3:** Label the vertices of \( R \) with \( a + q + \frac{n+1}{2} + 1, a + q + \frac{n+1}{2} + 3, \ldots, a + q + \frac{n+1}{2} + n - 2 \).

  The corresponding edge labels are \( 2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4 \).

  To show no duplication of vertex labels, notice that we need \( a + q - \frac{n+1}{2} + n - 2 < a + q + \frac{n+1}{2} + 1 \) which simplifies to \( n - 2 < n + 2 \) which is always true. Hence there are no duplicate vertex labels in the \( P_n \) component.
• **Case 2: \( n \equiv 2 \pmod{4} \)
  
  - **Case 2.a: \( a + q \equiv 0 \pmod{2} \)**
    
    *Step 2:* Label the vertices of \( L \) with \( a + q - \frac{n}{2}, a + q - \frac{n}{2} + 2, \ldots, a + q - \frac{n}{2} + n - 2 \).

    *Step 3:* Label the vertices of \( R \) with \( a + q + \frac{n}{2}, a + q + \frac{n}{2} + 2, \ldots, a + q + \frac{n}{2} + n - 2 \).

    The corresponding edge labels are \( 2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4 \) as shown in Figure 3.

To show no duplication of vertex labels, notice that we need \( a + q - \frac{n}{2} + n - 2 < a + q + \frac{n}{2} \) which simplifies to \( n - 2 < n \) which is always true. Hence there are no duplicate vertex labels in the \( P_n \) component.

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**Figure 3:** \( P_6 \) vertex labeling for \( 2a = 6, q = 10, \) modulo 30, Theorem 3.2, Case 2.b

- **Case 2.b: \( a + q \equiv 1 \pmod{2} \)**
  
  *Step 2:* Label the vertices of \( L \) with \( a + q - \frac{n}{2} - 1, a + q - \frac{n}{2} + 1, \ldots, a + q - \frac{n}{2} + n - 3 \).

  *Step 3:* Label the vertices of \( R \) with \( a + q + \frac{n}{2} + 1, a + q + \frac{n}{2} + 3, \ldots, a + q + \frac{n}{2} + n - 1 \).

  The corresponding edge labels are \( 2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4 \) as shown in Figure 3.
To show no duplication of vertex labels, notice that we need $a + q - \frac{n}{2} + n - 3 < a + q + \frac{n}{2} + 1$ which simplifies to $n - 3 < n + 1$ which is always true. Hence there are no duplicate vertex labels in the $P_n$ component.

\[ \square \]

**Definition 3.1.** We call a graph $G$ pseudo-bipartite if $G$ is not bipartite but the removal of one edge of $G$ results in a bipartite graph. The two vertex sets of the resulting bipartite graph are called pseudo-bipartite sets.

The pseudo-bipartite graphs of interest to us are odd cycles with pendant edges. We use $C_{m}^{+n}(l, r)$ to denote an $m$-cycle with $n$ pendant edges, where $l$ is the number of vertices in the left pseudo-bipartite set $L$ and $r$ is the number of vertices in the right pseudo-bipartite set $R$.

**Theorem 3.3.** $G \cup C_{m}^{+n}(l, r)$ is properly even harmonious if $G$ is an even $2a$-sequential graph when $m$ is odd.

**Proof.** When $G$ has $q$ edges the modulus is $2q + 2m + 2n$. Label the vertices of $G$ using the $2a$-sequential labeling. Draw $C_{m}^{+n}(l, r)$ zigzagging between the two pseudo-bipartite sets as shown in Figure 4.

**Step 1:** Label the vertices of $L$ with $a + q - l + 1, a + q - l + 3, \ldots, a + q + l - 1$.

**Step 2:** Label the vertices of $R$ with $a + q + l + 1, a + q + l + 3, \ldots, a + q + l + 2r - 1$.

**Figure 4:** $C_{7}^{+2}(5, 4)$ vertex labeling for $2a = 6$, $q = 6$, modulo 30, Theorem 3.3
The corresponding edge labels are \(2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2l + 2r - 2 = 2a + 2q + 2m + 2n - 2\) as shown in Figure 4.

Since we may assume the vertex labels of \(G\) and the hairy cycle have opposite parity, there is no duplication between vertex labels of the two components. Notice that all vertex labels in the \(C_m^{+n}(l, r)\) component are odd sequential increasing; therefore, there are no duplicated vertex labels.

\[\text{Figure 5: } Cat_5^{+3}(5, 3) \text{ vertex labeling for } 2a = 10, \ q = 5, \ \text{modulo } 24, \ \text{Theorem 3.4}\]

(a) Original labeling

(b) After a shift of labels with \(k = 3\).

**Theorem 3.4.** \(G \cup Cat_m^{+n}(l, r)\) is properly even harmonious if \(G\) is an even 2a-sequential and \(m > 1\).

*Proof.* When \(G\) has \(q\) edges the modulus is \(2q + 2m + 2n - 2\). Label the vertices of \(G\) using the 2a-sequential labeling.

**Step 1:** Arrange the caterpillar into a bipartite set. Denote the set on the left as \(L\) and the set on the right as \(R\) where \(|L| = l\) and \(|R| = r\) and \(l \geq r\).

**Step 2:** Label \(L\) with \(1, 3, \ldots, 2l - 1\) and label \(R\) with \(2a + 2q - 1, 2a + 2q + 1, \ldots, 2a + 2q + 2r - 3\).

The corresponding edge labels are \(2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2l + 2r - 4\).

If there is a duplication of vertex labels for this labeling for \(R\) and the number \(k\) of vertex label duplicates is even, subtract \(k\) from all vertex labels.
in $L$ and add $k$ to all vertex labels in $R$. The edges will have the same labeling as described previously. On the other hand, if the number $k$ of vertex label duplicates is odd, subtract $k + 1$ from all vertex labels in $L$ and add $k + 1$ to all vertex labels in $R$. The edges will have the same labeling as described previously. Figure 5 shows the repeated labeling for this caterpillar and the labeling after the appropriate adjustment.

Since we may assume the vertex labels of $G$ have even parity, there is no duplication between vertex labels in $G$ and vertex labels in the caterpillar. By shifting appropriately, it is clear that there are no vertex label duplications in the caterpillar component.

We use $W_{2n+1}$ to denote the wheel $C_{2n+1} + K_1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{$W_{11}$ vertex labeling for $2a = 8$, $q = 10$, $n = 5$, modulo 64, Theorem 3.5}
\end{figure}

**Theorem 3.5.** $G \cup W_{2n+1}$ is properly even harmonious if $G$ is an even $2a$-sequential graph.

**Proof.** When $G$ has $q$ edges the modulus is $2q + 8n + 4$. Label the vertices of $G$ using the $2a$-sequential labeling.

**Step 1:** Label every other vertex of the $2n+1$ cycle by starting with $a + q - n$, and incrementing by 2 each time. The last vertex will have the label $a + q + n$. The vertex labels have the same parity and the corresponding edges are the even integers from $2a + 2q$ through $2a + 2q + 4n$.

**Step 2:** Label the middle vertex of $W_{2n+1}$ as $a + q + 5n + 2$. This will pick up the edge labels from $2a + 2q + 4n + 2$ through $2a + 2q + 8n + 2 = (2a - 2) \pmod{2q + 8n + 4}$ as seen in Figure 6.
Since we may assume the parity of the vertex labels of $G$ is the opposite of the parity of the labels for $W_{2n+1}$, there is no duplication between vertex labels in $G$ and vertex labels in $W_{2n+1}$. The edge labels of $W_{2n+1}$ form an arithmetic progression of common difference two starting with $2a + 2q$ and going to $2a - 2$, therefore this labeling is properly even harmonious. □

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References


