PROPERLY EVEN HARMONIOUS LABELINGS OF DISJOINT UNIONS WITH EVEN SEQUENTIAL GRAPHS

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Abstract

A graph \( G \) with \( q \) edges is said to be harmonious if there exists an injection \( f \) from the vertices of \( G \) to the group of integers modulo \( q \) such that when each edge \( xy \) is assigned the label \( f(x) + f(y) \pmod{q} \), the resulting edge labels are distinct. If \( G \) is a tree, exactly one label may be used on two vertices. Over the years, many variations of harmonious labelings have been introduced.

An injective labeling \( f \) of a graph \( G \) with \( q \) edges is even 2a-sequential if the vertex labels are from \( \{0, 1, \ldots, 2q-1\} \) and the edge labels induced by \( f(u) + f(v) \) for each edge \( uv \) are \( 2a, 2a+2, \ldots, 2a+2q-2 \). When \( G \) is a tree, the allowable vertex labels are \( 0, 1, \ldots, 2q \).

We study a variant of harmonious labeling. A function \( f \) is properly even harmonious labeling of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the integers from 0 to \( 2q-2 \) and the induced function \( f^* \) from the edges of \( G \) to \( 0, 2, \ldots, 2(q-1) \) defined by \( f^*(xy) = f(x) + f(y)(\text{mod } 2q) \) is bijective.

This paper focuses on finding even harmonious labelings for disjoint graphs where one of these graphs has a 2a-sequential labeling. Among the families we investigate are: the disjoint union of 2a-sequential graphs with paths, the square of paths, caterpillars, and wheels.

1 Introduction

A vertex labeling of a graph \( G \) is a mapping \( f \) from the vertices of \( G \) to a set of elements, often integers. Each edge \( xy \) has a label that depends on adjacent vertices \( x \) and \( y \) and their labels \( f(x) \) and \( f(y) \). Graph labeling methods began with Rosa [7] in 1967. In 1980, Graham and Sloane [5] introduced harmonious labelings in connection with error-correcting codes and channel assignment problems. There have been three published papers regarding even harmonious graph labelings by Sarasija and Binthiya [8,9] and Gallian and Shoenhard [2]. An extensive survey of graph labeling methods is available online [1]. We follow the notation in [1].

2 Preliminaries

In this section we provide definitions and our motivation for focusing on disconnected graphs.

**Definition 2.1.** A graph \( G \) with \( q \) edges is said to be harmonious if there exists an injection \( f \) from the vertices of \( G \) to the group of integers modulo \( q \) such that when each edge \( xy \) is assigned the label \( f(x) + f(y)(\text{mod } q) \), the resulting
edge labels are distinct. When $G$ is a tree, exactly one edge label may be used on two vertices.

**Definition 2.2.** A function $f$ is said to be an *even harmonious* labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q$ and the induced function $f^*$ from the edges of $G$ to $0, 2, \ldots, 2(q_1)$ defined by $f^*(xy) = f(x) + f(y) (\text{mod } 2q)$ is bijective.

**Definition 2.3.** An even harmonious labeling of a graph $G$ with $q$ edges is said to be a *properly even harmonious labeling* if the vertex labels belong to \(\{0, 2, \ldots, 2q - 2\}\).

Because 0 and $2q$ are equal modulo $2q$, Gallian and Schoenhard [2] introduced the following more desirable form of even harmonious labelings.

**Definition 2.4.** A graph that has a (properly) even harmonious labeling is called a (properly) *even harmonious* graph.

**Definition 2.5.** [3] An injective labeling $f$ of a graph $G$ with $q$ edges is *a-sequential* if the vertex labels are from $\{0, 1, \ldots, q - 1\}$ and the edge labels induced by $f(u) + f(v)$ for each edge $uv$ are $a, \ldots, a + q - 1$. When $G$ is a tree, the allowable vertex labels are $0, 1, \ldots, q$.

**Definition 2.6.** An injective labeling $f$ of a graph $G$ with $q$ edges is *even 2a-sequential* if the vertex labels are from $\{0, 1, \ldots, 2q - 1\}$ and the edge labels induced by $f(u) + f(v)$ for each edge $uv$ are $2a, 2a + 2, \ldots, 2a + 2q - 2$. When $G$ is a tree, the allowable vertex labels are $0, 1, \ldots, 2q$.

By taking the edge labels of an $a$-sequentially labeled graph with $q$ edges modulo $q$, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Numerous families of $a$-sequential graphs are listed in [1].

For connected $a$-sequential graphs, a connected $2a$-sequential graph can be obtained by multiplying all the vertex labels by 2. Notice that the vertex labels in resulting graph belong to $\{0, 2, \ldots, 2q - 2\}$ (or $\{0, 2, \ldots, 2q\}$ for trees) and the edges labels are from $2a$ to $2a + 2q - 2$. Moreover, a connected $a$-sequential graph can be obtained from a connected even $2a$-sequential graph with even vertex labels by dividing all the vertex labels by 2. Likewise, a $2a$-sequential labeling of a connected graph with odd vertex labels induces an $a$-sequential labeling of the graph by subtracting 1 from each vertex label and dividing by 2. Thus for connected graphs, $a$-sequential is equivalent to $2a$-sequential. Consequently, we focus on even harmonious labelings of disconnected graphs.

### 3 Disconnected Graphs

In this section we give properly even harmonious labelings for the disjoint unions of a $2a$-sequential graph and paths, the square of paths, caterpillars, and wheels.
Theorem 3.1. $G \cup P_n^2$ is properly even harmonious if $G$ is an even $2a$-sequential graph and $n > 2$.

Proof. When $G$ has $q$ edges the modulus is $2q + 4n - 6$. Label the vertices of $G$ using the $2a$-sequential labeling. Label the vertices of $P_n^2$ with $a + q - 1, a + q + 1, \ldots, a + q + 2n - 3$ as shown in Figure 1. The corresponding edge labels are the even integers from $2a + 2q$ to $2a + 2q + 4n - 8$.

Since we may assume that the vertex labels of $G$ and $P_n^2$ have opposite parity, their labels have nothing in common. Moreover, because the vertex labels in the $P_n^2$ component are increasing and the gap between the largest vertex label and smallest vertex label of $P_n^2$ is $2n - 2$ and the modulus is $2q + 4n - 6$ there is no wrap around to the first vertex label of $P_n^2$.

Figure 2: $P_n^2$ vertex labeling for $2a = 8$, $q = 10$, modulo 36, Theorem 3.2, Case 1.a
Theorem 3.1 raises the question that if $G$ is an even $2a$-sequential graph can we find a properly even harmonious labeling for $G \cup P_n$ given some conditions on $n$?

**Theorem 3.2.** $G \cup P_n$ is properly even harmonious if $G$ is an even $2a$-sequential graph when $n > 1$ and $n \equiv 1, 2 \pmod{4}$.

**Proof.** When $G$ has $q$ edges the modulus is $2q + 2n - 2$. Label the vertices of $G$ using the $2a$-sequential labeling.

*Step 1:* Arrange $P_n$ into a bipartite set; denote the set on the left $L$ and the set on the right $R$. (See Figure 3.)

- **Case 1:** $n \equiv 1 \pmod{4}$

  - **Case 1.a:** \( a + q \equiv 0 \pmod{2} \).

    **Step 2:** Label the vertices of $L$ with \( a + q - \frac{n+1}{2}, a + q - \frac{n+1}{2} + 2, \ldots, a + q - \frac{n+1}{2} + n - 1 \).

    **Step 3:** Label the vertices of $R$ with \( a + q + \frac{n+1}{2}, a + q + \frac{n+1}{2} + 2, \ldots, a + q + \frac{n+1}{2} + n - 3 \).

    The corresponding edge labels are \( 2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4 \) as shown in Figure 2.

    To show no duplication of vertex labels, notice that we need \( a + q - \frac{n+1}{2} + n - 1 < a + q + \frac{n+1}{2} \) which simplifies to \( n - 1 < n + 1 \) which is always true. Hence there are no duplicate vertex labels in the $P_n$ component.

  - **Case 1.b:** \( a + q \equiv 1 \pmod{2} \)

    **Step 2:** Label the vertices of $L$ with \( a + q - \frac{n+1}{2} - 1, a + q - \frac{n+1}{2} + 1, \ldots, a + q - \frac{n+1}{2} + n - 2 \).

    **Step 3:** Label the vertices of $R$ with \( a + q + \frac{n+1}{2} + 1, a + q + \frac{n+1}{2} + 3, \ldots, a + q + \frac{n+1}{2} + n - 2 \).

    The corresponding edge labels are \( 2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4 \).

    To show no duplication of vertex labels, notice that we need \( a + q - \frac{n+1}{2} + n - 2 < a + q + \frac{n+1}{2} + 1 \) which simplifies to \( n - 2 < n + 2 \) which is always true. Hence there are no duplicate vertex labels in the $P_n$ component.
• **Case 2**: $n \equiv 2 \pmod{4}$

  - **Case 2.a**: $a + q \equiv 0 \pmod{2}$

    **Step 2**: Label the vertices of $L$ with $a + q - \frac{n}{2}, a + q - \frac{n}{2} + 2, \ldots, a + q - \frac{n}{2} + n - 2$.

    **Step 3**: Label the vertices of $R$ with $a + q + \frac{n}{2}, a + q + \frac{n}{2} + 2, \ldots, a + q + \frac{n}{2} + n - 2$.

    The corresponding edge labels are $2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4$.

    To show no duplication of vertex labels, notice that we need $a + q - \frac{n}{2} + n - 2 < a + q + \frac{n}{2}$ which simplifies to $n - 2 < n$ which is always true. Hence there are no duplicate vertex labels in the $P_n$ component.

![Figure 3: $P_6$ vertex labeling for $2a = 6$, $q = 10$, modulo 30, Theorem 3.2, Case 2.b](image)

- **Case 2.b**: $a + q \equiv 1 \pmod{2}$

  **Step 2**: Label the vertices of $L$ with $a + q - \frac{n}{2} - 1, a + q - \frac{n}{2} + 1, \ldots, a + q - \frac{n}{2} + n - 3$.

  **Step 3**: Label the vertices of $R$ with $a + q + \frac{n}{2} + 1, a + q + \frac{n}{2} + 3, \ldots, a + q + \frac{n}{2} + n - 1$.

  The corresponding edge labels are $2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2n - 4$ as shown in Figure 3.
To show no duplication of vertex labels, notice that we need \( a + q - \frac{n}{2} + n - 3 < a + q + \frac{n}{2} + 1 \) which simplifies to \( n - 3 < n + 1 \) which is always true. Hence there are no duplicate vertex labels in the \( P_n \) component.

\[ \Box \]

**Definition 3.1.** We call a graph \( G \) pseudo-bipartite if \( G \) is not bipartite but the removal of one edge of \( G \) results in a bipartite graph. The two vertex sets of the resulting bipartite graph are called pseudo-bipartite sets.

The pseudo-bipartite graphs of interest to us are odd cycles with pendant edges. We use \( C_{m+n}^{+l}(l, r) \) to denote an \( m \)-cycle with \( n \) pendant edges, where \( l \) is the number of vertices in the left pseudo-bipartite set \( L \) and \( r \) is the number of vertices in the right pseudo-bipartite set \( R \).

![Figure 4: \( C_{m+n}^{+l}(l, r) \) vertex labeling for \( 2a = 6, q = 6 \), modulo 30, Theorem 3.3](image)

**Theorem 3.3.** \( G \cup C_{m+n}^{+l}(l, r) \) is properly even harmonious if \( G \) is an even \( 2a \)-sequential graph when \( m \) is odd.

**Proof.** When \( G \) has \( q \) edges the modulus is \( 2q + 2m + 2n \). Label the vertices of \( G \) using the \( 2a \)-sequential labeling. Draw \( C_{m+n}^{+l}(l, r) \) zigzagging between the two pseudo-bipartite sets as shown in Figure 4.

**Step 1:** Label the vertices of \( L \) with \( a + q - l + 1, a + q - l + 3, \ldots, a + q + l - 1 \).

**Step 2:** Label the vertices of \( R \) with \( a + q + l + 1, a + q + l + 3, \ldots, a + q + l + 2r - 1 \).
The corresponding edge labels are $2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2l + 2r - 2 = 2a + 2q + 2m + 2n - 2$ as shown in Figure 4.

Since we may assume the vertex labels of $G$ and the hairy cycle have opposite parity, there is no duplication between vertex labels of the two components. Notice that all vertex labels in the $C_m^n(l, r)$ component are odd sequential increasing; therefore, there are no duplicated vertex labels. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Cat$_5^+3(5, 3)$ vertex labeling for $2a = 10$, $q = 5$, modulo 24, Theorem 3.4}
\end{figure}

(a) Original labeling
(b) After a shift of labels with $k = 3$.

**Theorem 3.4.** $G \cup \text{Cat}_m^n(l, r)$ is properly even harmonious if $G$ is an even 2$a$-sequential and $m > 1$.

**Proof.** When $G$ has $q$ edges the modulus is $2q + 2m + 2n - 2$. Label the vertices of $G$ using the 2$a$-sequential labeling.

1. **Step 1:** Arrange the caterpillar into a bipartite set. Denote the set on the left as $L$ and the set on the right as $R$ where $|L| = l$ and $|R| = r$ and $l \geq r$.

2. **Step 2:** Label $L$ with $1, 3, \ldots, 2l - 1$ and label $R$ with $2a + 2q - 1, 2a + 2q + 1, \ldots, 2a + 2q + 2r - 3$.

The corresponding edge labels are $2a + 2q, 2a + 2q + 2, \ldots, 2a + 2q + 2l + 2r - 4$.

If there is a duplication of vertex labels for this labeling for $R$ and the number $k$ of vertex label duplicates is even, subtract $k$ from all vertex labels.
in $L$ and add $k$ to all vertex labels in $R$. The edges will have the same labeling as described previously. On the other hand, if the number $k$ of vertex label duplicates is odd, subtract $k + 1$ from all vertex labels in $L$ and add $k + 1$ to all vertex labels in $R$. The edges will have the same labeling as described previously. Figure 5 shows the repeated labeling for this caterpillar and the labeling after the appropriate adjustment.

Since we may assume the vertex labels of $G$ have even parity, there is no duplication between vertex labels in $G$ and vertex labels in the caterpillar. By shifting appropriately, it is clear that there are no vertex label duplications in the caterpillar component. □

We use $W_{2n+1}$ to denote the wheel $C_{2n+1} + K_1$.

![Figure 6: $W_{11}$ vertex labeling for $2a = 8$, $q = 10$, $n = 5$, modulo 64, Theorem 3.5](image)

**Theorem 3.5.** $G \cup W_{2n+1}$ is properly even harmonious if $G$ is an even $2a$-sequential graph.

**Proof.** When $G$ has $q$ edges the modulus is $2q + 8n + 4$. Label the vertices of $G$ using the $2a$-sequential labeling.

**Step 1:** Label every other vertex of the $2n + 1$ cycle by starting with $a + q - n$, and incrementing by 2 each time. The last vertex will have the label $a + q + n$. The vertex labels have the same parity and the corresponding edges are the even integers from $2a + 2q$ through $2a + 2q + 4n$.

**Step 2:** Label the middle vertex of $W_{2n+1}$ as $a + q + 5n + 2$. This will pick up the edge labels from $2a + 2q + 4n + 2$ through $2a + 2q + 8n + 2 = (2a - 2) \ (mod \ 2q + 8n + 4)$ as seen in Figure 6.
Since we may assume the parity of the vertex labels of $G$ is the opposite of the parity of the labels for $W_{2n+1}$, there is no duplication between vertex labels in $G$ and vertex labels in $W_{2n+1}$. The edge labels of $W_{2n+1}$ form an arithmetic progression of common difference two starting with $2a + 2q$ and going to $2a - 2$, therefore this labeling is properly even harmonious. 

\[ \square \]

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**References**


