ON THE EDGE-BALANCED INDEX SETS OF COMPLETE EVEN BIPARTITE GRAPHS

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Abstract

In 2009, Kong, Wang, and Lee introduced the problem of finding the edge-balanced index sets (EBI) of complete bipartite graphs $K_{m,n}$ by examining the cases where $m \geq n$ and $n = 1, 2, 3, 4,$ and $5$, as well as the case where $m = n \geq 6$. Since then, the problem of finding $EBI(K_{m,n})$ has been solved in the case where $m \geq n \geq 1$ are both odd and in the case where $m$ is odd, $n$ is even, and $m > n$. In this paper, we find the edge-balanced index sets for complete even bipartite graphs. That is, we solve the $EBI(K_{m,n})$ problem in the case where $m \geq n \geq 2$ are both even.

1 Introduction

Given a graph $G$, let $V$ and $E$ denote the vertex set and edge set of $G$, respectively. A binary edge-labeling of $G$ is a surjective function $f : E \to \{0,1\}$. For $i \in \{0,1\}$, we call $e \in E$ an $i$-edge if $f(e) = i$. Let $e(i)$ denote the number of $i$-edges under a binary edge-labeling $f$. If $|e(1) - e(0)| \leq 1$, we say that $f$ is edge-friendly. Under $f$, we let $\deg_i(v)$ denote the number of $i$-edges incident with $v \in V$, and note that $\deg(v) = \deg_0(v) + \deg_1(v)$. A partial function $g$ from $X$ to $Y$ is a function from $A$ to $Y$ for some subset $A$ of $X$, where $g(x)$ is undefined for $x \in X$ and $x \notin A$ [5]. If $f$ is edge-friendly and $v$ is a vertex, then $f$ will induce a partial vertex-labeling $f^*$ where $f^*(v) = 1$ if $\deg_1(v) > \deg_0(v)$ and $f^*(v) = 0$ if $\deg_0(v) > \deg_1(v)$. If $\deg_1(v) = \deg_0(v)$, then $f^*(v)$ is undefined. For $i \in \{0,1\}$, if $f^*(v) = i$, then $v$ is called an $i$-vertex. If $f^*(v)$ is undefined, we will say that $v$ is unlabeled. That is, an unlabeled vertex is a vertex that is not labeled 0 or 1 under $f^*$. Let $v(0)$ and $v(1)$ denote the number of 0- and 1-vertices, respectively, under $f^*$. The edge-balanced index set of $G$ is defined as

$$EBI(G) = \{|v(1) - v(0)| : f \text{ is edge-friendly}\}.$$ 

An element in $EBI(G)$ is called a balanced index. Of course, if $f$ is an edge-friendly labeling and $v(0) \geq v(1)$ under the induced vertex-labeling, then the complementary edge-friendly labeling $f_c$, which switches the labels on the 0- and 1-edges under $f$ simultaneously, will induce a vertex-labeling where $v(1) \geq v(0)$. Then, without loss of generality, we may assume that edge-friendly labelings are chosen so that $v(1) \geq v(0)$. With this assumption, all balanced indices can be computed simply as $v(1) - v(0)$.

Motivated by a lack of proof for the Graceful Tree Conjecture, which claims that all tree are graceful (see [1,10]), Cahit introduced cordial labelings, a weaker version of graceful labelings [2]. A cordial labeling of a graph $G$ is a binary vertex-labeling $f : V \to \{0,1\}$, where $|v(1) - v(0)| \leq 1$, that induces a binary edge-labeling $f : E \to \{0,1\}$,
where \( \overline{f}(uv) = |f(u) - f(v)|, \) such that \( |e(1) - e(0)| \leq 1. \) In [11], Yilmaz and Cahit define an \( E \)-cordial labeling, a weaker version of an edge-graceful labeling. An \( E \)-cordial labeling is a binary edge-labeling \( f : E \to \{0, 1\}, \) where \( |e(1) - e(0)| \leq 1, \) that induces a binary vertex-labeling \( \overline{f} : V \to \{0, 1\}, \) where \( \overline{f}(v) = \sum_{uv \in E} f(uv) \mod 2, \) such that \( |\overline{v}(1) - \overline{v}(0)| \leq 1. \) Both edge-friendly and \( E \)-cordial labelings meet the requirement that \( |e(1) - e(0)| \leq 1; \) that is, they both satisfy a cordial condition over the edges. However, for an \( E \)-cordial labeling, it is required that \( |\overline{v}(1) - \overline{v}(0)| \leq 1, \) whereas an edge-friendly labeling has no such restriction on the number of 0- and 1-vertices. Indeed, an edge-balanced index set problem is one that explores the variance in the quantities \( v(1) \) and \( v(0). \) More information about graph labelings can be found in Gallian’s dynamic survey [4].

The edge-balanced index set problem has a relatively short history when compared to that of the Graceful Tree Conjecture. In 2009, three articles related to the \( EBI_1 \) problem appeared. Kwong and Lee and Chopra, Lee, and Su provided the initial results in the field in their respective studies of the edge-balanced index sets of generalized theta graphs [9] and fans and broken fans [3]. In [7], Kong, Wang, and Lee found the edge-balanced index sets of complete bipartite graphs \( K_{m,n} \) in the cases where \( m \geq n \) and \( n = 1, 2, 3, 4, \) and 5, as well as the case where \( m = n \geq 6. \) Note that because \( K_{m,n} \) and \( K_{n,m} \) are isomorphic, the cases where \( m \geq n \) (and \( m \) and \( n \) have the same parity) or \( m > n \) (and \( m \) and \( n \) are of opposite parity) need only be considered. In two papers published in 2014, the more general \( EBI(K_{m,n}) \) problem was addressed. In [8], Krop et al. found the edge-balanced index sets for complete bipartite graphs where both partite sets are of odd cardinality. In [6], Hua and Raridan determined the edge-balanced index sets for complete bipartite graphs where the larger partite set is of odd cardinality and the smaller is of even cardinality. In this paper, we find the edge-balanced index sets for complete bipartite graphs where both partite sets have even cardinality.

2 Notation

Throughout the rest of this paper, we let \( K_{m,n} \) denote a complete even bipartite graph; that is, \( m \) and \( n \) are both even and \( m \geq n \geq 2. \) The vertex set of \( K_{m,n} \) is partitioned into two disjoint nonempty subsets \( A = \{v_1, v_2, \ldots, v_m\} \) and \( B = \{u_1, u_2, \ldots, u_n\}. \) For any edge-friendly labeling of \( K_{m,n}, \) we have that \( e(0) = e(1) = \frac{mn}{2} \) since the total number of edges, \( mn, \) is even.

For integers \( a < b, \) define \([a, b] = \{a, a + 1, \ldots, b\}\) and \([a, a] = \{a\}. \) If \( a \) is positive, \([a] = \{1, 2, \ldots, a\}; \) otherwise, \([a] \) is the empty set. We organize the edge-labels of an edge-friendly labeling of \( K_{m,n} \) as an \( n \times m \) binary matrix whose \((i,j)\)-entry, \( a_{ij}, \) is the label on edge \( u_i v_j, \) where \( i \in [n] \) and \( j \in [m]. \) Note that for \( j \in [m], \) \( \deg_1(v_j) = \sum_{i=1}^{n} a_{ij} \) is the sum of the entries in column \( j. \) For an edge-friendly labeling of \( K_{m,n}, \) every edge is labeled 0 or 1, so \( \deg_0(v_j) = \deg(v_j) - \deg_1(v_j) = n - \sum_{i=1}^{n} a_{ij}. \) For each vertex \( v_j \in A, \) it follows that if the sum of the entries in column \( j \) is less than \( \frac{n}{2}, \) then vertex \( v_j \) is a 0-vertex;
if the sum of the entries in column \( j \) is \( \frac{n}{2} \), then \( v_j \) is unlabeled; and, if the sum of the entries in column \( j \) is greater than \( \frac{n}{2} \), then \( v_j \) is a 1-vertex. Since \( \deg_1(u_i) = \sum_{j=1}^{m} a_{ij} \) and \( \deg_0(u_i) = m - \sum_{j=1}^{m} a_{ij} \) for \( i \in [n] \), similar results follow for vertices \( u_i \in B \) by replacing “column \( j \)” with “row \( i \)” and the quantity \( \frac{n}{2} \) with \( \frac{m}{2} \). That is, organizing the edge-labels of an edge-friendly labeling of \( K_{m,2} \) into an \( n \times m \) matrix allows us to determine if a vertex in \( A \) or \( B \) is a 0-vertex, a 1-vertex, or is unlabeled by simply comparing each column sum to \( \frac{n}{2} \) or each row sum to \( \frac{m}{2} \), respectively.

**Example 2.1.** Finding the corresponding balanced index for each of the following edge-friendly labelings of \( K_{4,4} \) is straightforward. Figures 1(a) and 1(b) show two different edge-friendly labelings that each give \( 0 \in \text{EBI}(K_{4,4}) \). In particular, from the matrix in Figure 1(a), we determine that \( A \) contains only unlabeled vertices and \( B \) contains two 1-vertices and two 0-vertices. Hence, \( v(1) = v(0) = 2 \). Figure 1(c) shows that \( 1 \in \text{EBI}(K_{4,4}) \) and Figure 1(d) shows that \( 2 \in \text{EBI}(K_{4,4}) \).

![Figure 1: Some edge-friendly labelings of \( K_{4,4} \).](image)

The Division Algorithm states that for any integer \( x \) and positive integer \( y \), there exist unique integers \( q \) and \( r \) such that \( x = qy + r \) and \( 0 \leq r < y \). We denote the quotient \( q \) by \( x \div y \) (or by \( \lfloor \frac{x}{y} \rfloor \)) and the remainder \( r \) by \( x \mod y \).

### 3 Finding \( \text{EBI}(K_{m,n}) \)

In this section, we prove our main result.

**Theorem 3.1.** Let \( K_{m,n} \) be a complete bipartite graph with partite sets of even cardinality \( m \) and \( n \), where \( m \geq n \). Then \( \text{EBI}(K_{m,2}) = \{0\} \). For \( n \geq 4 \), let \( k = \lfloor \frac{mn}{n+2} \rfloor \), \( k' = \frac{mn}{2} \mod \left( \frac{n}{2} + 1 \right) \), \( j = \lfloor \frac{mn}{n+2} \rfloor \), and \( j' = \frac{mn}{2} \mod \left( \frac{n}{2} + 1 \right) \). Then

\[
\text{EBI}(K_{m,n}) = \begin{cases} 
\{0,1,\ldots,2(k+j)+2-m-n\}, & \text{if } k' = \frac{n}{2} \text{ and } j' = \frac{m}{2}, \\
\{0,1,\ldots,2(k+j)+1-m-n\}, & \text{if } \text{either } k' = \frac{n}{2} \text{ or } j' = \frac{m}{2}, \\
\{0,1,\ldots,2(k+j)-m-n\}, & \text{if } k' < \frac{n}{2} \text{ and } j' < \frac{m}{2}.
\end{cases}
\]

**Proof.** In [7], the authors show that \( \text{EBI}(K_{m,2}) = \{0\} \) for all integers \( m \geq 2 \). Throughout the rest of this proof, we assume that \( m \) and \( n \) are both even and \( m \geq n \geq 4 \).
To find the maximal element of $EBI(K_{m,n})$, we need an edge-friendly labeling that maximizes the value of $v(1)$ while at the same time minimizes the value of $v(0)$. For $i \in \{0, 1\}$, let $v_A(i)$ and $v_B(i)$ represent the number of $i$-vertices in $A$ and $B$, respectively, and note that $v(i) = v_A(i) + v_B(i)$. Let $k$ and $j$ represent the maximum value of $v_A(1)$ and $v_B(1)$, respectively. A 1-vertex $v \in A$ must have $\deg_1(v) \geq \frac{n}{2} + 1$, so the maximum value of $v_A(1)$ is $k = e(1) \mod \left( \frac{n}{2} + 1 \right) = \left\lfloor \frac{mn}{m+n+2} \right\rfloor$. Given any edge-friendly labeling that maximizes $v_A(1)$ where each of the 1-vertices in $A$ is incident with exactly $(\frac{n}{2} + 1)$ 1-edges, the number of 1-edges incident with the other $(m - k)$ vertices in $A$ is $k' = e(1) \mod \left( \frac{n}{2} + 1 \right)$. If $v_A(1)$ is maximized and $k' < \frac{n}{2}$, there are not enough of these “extra” 1-edges to allow $A$ to contain an unlabeled vertex, so $v_A(0) = m - k$. On the other hand, if $v_A(1)$ is maximized and $k' = \frac{n}{2}$, there are enough extra 1-edges to allow an unlabeled vertex in $A$, and having an unlabeled vertex in $A$ reduces the value of $v_A(0)$, which would be $m - k - 1$ in this case. Similarly, the maximum value of $v_B(1)$ is $j = e(1) \mod \left( \frac{n}{2} + 1 \right) = \left\lfloor \frac{mn}{m+n+2} \right\rfloor$. For any edge-friendly labeling that maximizes $v_B(1)$ where each of the 1-vertices in $B$ has exactly $(\frac{n}{2} + 1)$ 1-edges, the number of 1-edges incident with the other $(n - j)$ vertices in $B$ is $j' = e(1) \mod \left( \frac{n}{2} + 1 \right)$. When $v_B(1)$ is maximized, if $j' < \frac{n}{2}$, then $v_B(0) = n - j$, and if $j' = \frac{n}{2}$, then $v_B(0) = n - j - 1$.

Now, we need to find an edge-friendly labeling that maximizes both $v_A(1)$ and $v_B(1)$ at the same time, thus maximizing their sum $v(1)$. Maximizing $v(1)$ and allowing $A$ or $B$ to contain an unlabeled vertex (when $k' = \frac{n}{2}$ or $j' = \frac{n}{2}$, respectively) minimizes both $v_A(0)$ and $v_B(0)$ at the same time, thus minimizing their sum $v(0)$. That is,

$$\max EBI(K_{m,n}) = \begin{cases} 2(k+j) + 2 - m - n, & \text{if } k' = \frac{n}{2} \text{ and } j' = \frac{n}{2}, \\ 2(k+j) + 1 - m - n, & \text{if either } k' = \frac{n}{2} \text{ or } j' = \frac{n}{2}, \\ 2(k+j) - m - n, & \text{if } k' < \frac{n}{2} \text{ and } j' < \frac{n}{2}. \end{cases}$$

In the case where $m = n = 4$, we find that $k = j = k' = j' = \frac{n}{2} = \frac{4}{2} = 2$ and $\max EBI(K_{4,4}) = 2$. Example 2.1(d) shows an edge-friendly labeling for $K_{4,4}$ that produces the maximal balanced index for this graph. For all other values of even integers $m \geq n$, it follows that $k > \frac{n}{2}$ and $j > \frac{n}{2}$, which ensures that in each of the three cases above, $\max EBI(K_{m,n})$ is a positive quantity.

We now discuss an algorithm that provides a sequence of edge-friendly labelings (more specifically, a sequence of edge-label switches, each of which provides a different edge-friendly labeling) that correspond to each of the balanced indices from 0 to $\max EBI(K_{m,n})$. For each of the following steps, we mention only the vertices whose labels change due to the edge-label switches described in that step. For some values of $m$ and $n$, running the entire algorithm is unnecessary; indeed, the procedure should be terminated when $\max EBI(K_{m,n})$ has been obtained. We will provide a few example graphs when early termination is allowed.

**Step 0.** For $a \in \left[ \frac{n}{2} \right]$ and $b \in [m]$, create an $n \times m$ matrix whose $(a,b)$-entry is 1 and set all other entries to 0. The top half of this matrix is all 1s and the bottom half is all 0s.
Then $v_A(1) = v_A(0) = 0$ and $v_B(1) = v_B(0) = \frac{n}{2}$, which implies that $0 \in EBI(K_{m,n})$.

In Steps 1-3, we let $q(a)$ and $r(a)$ represent the quotient and remainder, respectively, when $(a - 1)$ is divided by $\frac{n}{2}$ using the Division Algorithm.

**Step 1.** For $a \in \left(\frac{m}{2} - 1, \frac{m}{2}\right]$, switch the $(\frac{m}{2} + 1, a)$-entry with the $(\frac{m}{2} - r(a), m - q(a))$-entry. Here, each edge-label switch exchanges a 0 in row $\frac{n}{2} + 1$ with a 1 in the last $q\left(\frac{m}{2} - 1\right) + 1$ columns and above the $(\frac{m}{2} + 1)$-st row. When all of these edge-label switches have been performed, vertex $v_a$ becomes a 1-vertex for $a \in \left(\frac{m}{2} - 1\right]$ and $v_{m+1-b}$ becomes a 0-vertex for $b \in \left[q\left(\frac{m}{2} - 1\right) + 1\right]$. Note that the first switch of a 1 in each column has no effect on the balanced index; otherwise, $v_A$ increase by 1, but that each subsequent switch will increase the balanced index by 1. At the end of Step 1, $\deg_0(u_{\frac{m}{2} + 1}) = \frac{m}{2} + 1$, which means vertex $u_{\frac{m}{2} + 1}$ is incident with exactly 2 more 0-edges than 1-edges. That is, any further edge-label switches that reduce the number 0-edges incident with $u_{\frac{m}{2} + 1}$ and increase the number incident 1-edges will cause the label on $u_{\frac{m}{2} + 1}$ to change, as we will see in the next step.

**Step 2.** Switch the $(\frac{m}{2} + 1, \frac{m}{2})$-entry with the $(\frac{m}{2} - r\left(\frac{m}{2}\right), \frac{m}{2})$-entry. This switch causes vertex $u_{\frac{m}{2} + 1}$ to become an unlabeled vertex so the balanced index increases by 1. Now, switch the $(\frac{m}{2} - r\left(\frac{m}{2}\right), \frac{m}{2})$-entry with the $(\frac{m}{2} - r\left(\frac{m}{2}\right), m - q\left(\frac{m}{2}\right))$-entry, which causes $v_{\frac{m}{2}}$ to become a 1-vertex. If $r\left(\frac{m}{2}\right) = 0$, or equivalently $m = tn + 2$ for some integer $t \geq 1$, then this switch also causes $v_{m-q\left(\frac{m}{2}\right)}$ to become a 0-vertex and there is no change in the balanced index; otherwise, $v_{m-q\left(\frac{m}{2}\right)}$ was already a 0-vertex and the balanced index increases by 1. Note that for $m = n = 4$, we terminate the procedure since $k = \frac{m}{2}$ and $j = \frac{n}{2}$ for this case.

For other values of $m$ and $n$, the $(\frac{m}{2} + 1)$-st (double) switch is similar to the $\frac{m}{2}$-th. Exchange the $(\frac{m}{2} + 1, \frac{m}{2} + 1)$-entry with the $(\frac{m}{2} - r\left(\frac{m}{2} + 1\right), \frac{m}{2} + 1)$-entry. This switch causes $u_{\frac{m}{2} + 1}$ to become a 1-vertex so the balanced index increases by 1. Now, switch the $(\frac{m}{2} - r\left(\frac{m}{2} + 1\right), \frac{m}{2} + 1)$-entry with the $(\frac{m}{2} - r\left(\frac{m}{2} + 1\right), m - q\left(\frac{m}{2} + 1\right))$-entry, which causes $v_{\frac{m}{2} + 1}$ to become a 1-vertex. If $r\left(\frac{m}{2} + 1\right) = 0$, or equivalently $m = tn$ for some integer $t \geq 1$, then this switch also causes $v_{m-q\left(\frac{m}{2} + 1\right)}$ to become a 0-vertex and there is no change in the balanced index. Otherwise, $v_{m-q\left(\frac{m}{2} + 1\right)}$ was already a 0-vertex and the balanced index increases by 1. Note that if $(m,n) = (6,4), (8,4), (10,4)$, or $(6,6)$, for example, then we terminate the procedure since $v_A(1) = k = \frac{m}{2} + 1, v_B(1) = j = \frac{n}{2} + 1$, and $j < \frac{m}{2}$. That is, maximizing $v_B(1)$ does not allow for an unlabeled vertex in $B$ for these cases.

**Step 3.** Perform this step if and only if $k > \frac{m}{2} + 1$. For $a \in \left[\frac{m}{2} + 2, k\right]$, switch the $(\frac{m}{2} + 1, a)$-entry with the $(\frac{m}{2} - r(a), m - q(a))$-entry. This step is essentially the same as Step 1, just applied to a different set of indices. Upon completing this step, we have now forced $v_A(1) = k$ and either $v_A(0) = m - k$ (there are no unlabeled vertices in $A$) or $v_A(0) = m - k - 1$ (there is one unlabeled vertex in $A$). That is, we have maximized
\(v_A(1)\) and minimized \(v_A(0)\). Additionally, \(v_B(1) = \frac{n}{2} + 1\) and \(v_B(0) = n - v_B(1)\), so if 
\(j = \frac{n}{2} + 1\) and \(j' < \frac{m}{2}\), then terminate the procedure.

**Step 4.** Perform this step if and only if \(j > \frac{n}{2} + 1\) or \(j' = \frac{m}{2}\). In this step, we only make switches with entries that are in the same column, thereby preserving the current vertex labels for all of the vertices in \(A\). Since Steps 1-3 exchange \(k\) 1s in the top half of the matrix with 0s from row \(\frac{n}{2} + 1\), we have \(\deg_1(u_s) = k + 1 > \frac{m}{2} + 2\) for \(s \in [c]\), where 
\(c = \frac{n}{2} - k \mod \frac{n}{2}\) is a positive integer. For these \(c\) 1-vertices in \(B\), we may replace up to \(d = (k + 1) - \left(\frac{n}{2} + 1\right) = k - \frac{n}{2} > 1\) of their incident 1-edges with 0-edges and the vertices will remain 1-vertices. Now, \(\deg_1(u_s) = k > \frac{m}{2} + 1\) for \(s \in \left[\frac{n}{2} + 1 - k \mod \frac{n}{2}, \frac{n}{2} + 1\right]\), so for these \((1 + k \mod \frac{n}{2})\) 1-vertices in \(B\), we may replace up to \(d - 1 > 0\) of their incident 1-edges with 0-edges and the vertices will remain 1-vertices. Moreover, \(\deg_1(u_s) = 0\) for \(s \in \left[\frac{n}{2} + 2, n\right]\).

Let \(b\) be the total number of switches that we need to perform to obtain the maximal balanced index. If \(j' = \frac{m}{2}\), then \(B\) should contain an unlabeled vertex and \(b = \frac{mn}{2} - \left(\frac{m}{2} + 1\right) \left(\frac{n}{2} + 1\right)\); otherwise, \(b = \left[j - \left(\frac{n}{2} + 1\right)\right] \left(\frac{m}{2} + 1\right)\). Recall that \(c = \frac{n}{2} - k \mod \frac{n}{2}\) and \(d = k - \frac{m}{2}\). For \(a \in [cd]\), switch the
\[
\left(1 + (a - 1) \div d, 1 + (a - 1) \mod \left(\frac{m}{2} + 1\right)\right)
\]
with the
\[
\left(\frac{n}{2} + 2 + (a - 1) \div \left(\frac{m}{2} + 1\right), 1 + (a - 1) \mod \left(\frac{m}{2} + 1\right)\right)
\]
This collection of switches takes the extra \(d\) 1s on row \(s\), where \(s \in [c]\), and exchanges them for 0s that are in the same column but on a different row (and in the bottom half of the matrix). The row that is losing 0s and gaining 1s will continue to do so until the number of 1s in that row is \(\frac{m}{2} + 1\), at which time the procedure simply moves down to the next row and “starts over” (due to the mod operator). When the number of 1s in a row reaches \(\frac{m}{2}\), the corresponding vertex changes from a 0-vertex to an unlabeled vertex and the balanced index increases by 1. Similarly, when the number of 1s in a row reaches \(\frac{m}{2} + 1\), the unlabeled vertex becomes a 1-vertex and the balanced index increases by 1 again.

Continuing, for \(a \in [cd + 1, b]\), switch the
\[
\left(1 + c + (a - 1 - cd) \div (d - 1), 1 + (a - 1) \mod \left(\frac{m}{2} + 1\right)\right)
\]
with the
\[
\left(\frac{n}{2} + 2 + (a - 1) \div \left(\frac{m}{2} + 1\right), 1 + (a - 1) \mod \left(\frac{m}{2} + 1\right)\right)
\]
This collection of switches is similar to those just completed, except we are taking only \(d - 1\) extra 1s on a row and exchanging them with 0s in the same column but on a different row. The balanced index changes as before, as well.
We started with balanced index 0 and every switch described by the algorithm increased the balanced index by at most 1. Although it is not required that every step of the algorithm be completed for all graphs $K_{m,n}$, any early termination of the procedure was due to having already obtained $\text{max} EBI(K_{m,n})$. At the end of Step 3, we remarked that $v_A(1) = k$ and either $v_A(0) = m - k - 1$ or $v_A(0) = m - k$, depending on whether $A$ does or does not contain an unlabeled vertex, respectively. Step 4 does not alter the labels of vertices in $A$. Upon completion of Step 4, we find that $v_B(1) = j$ and either $v_B(0) = n - j$ (there are no unlabeled vertices in $B$) or $v_B(0) = n - j - 1$ (there is one unlabeled vertex in $B$). Thus, the construction provided by the algorithm produces all of the balanced indices in $EBI(K_{m,n})$, from 0 up to and including $\text{max} EBI(K_{m,n})$.

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