ON HARMONIOUS COLORINGS OF REGULAR DIGRAPHS

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Abstract

Let $D$ be a directed graph with $n$ vertices and $m$ edges. A function $f : V(D) \rightarrow \{1, 2, 3, \ldots, k\}$ where $k \leq n$ is said to be harmonious coloring of $D$ if for any two edges $xy$ and $uv$ of $D$, the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If the pair $(i, i)$ is not assigned for any $i$, $1 \leq i \leq n$, then $f$ is said to be a proper harmonious coloring of $D$. The minimum $k$ for which $D$ admits a proper harmonious coloring is called the proper harmonious coloring number of $D$. We obtain the lower bound for proper harmonious coloring number of regular digraphs and investigate the same for oriented $n \times n$ torus and circulant digraph.

1 Introduction

Coloring the vertices and edges of a finite simple graph has often been motivated by their utility to various applied fields [10] and their mathematical interest. Various coloring problems such as the vertex coloring and edge coloring problem have been studied in the literature [5].

Definition 1.1. A coloring of a graph $G$ is a function $c : V(G) \rightarrow X$ for some set of colors $X$ such that $c(u) \neq c(v)$ for each edge $uv \in E(G)$.

The coloring defined above is the proper vertex coloring where we color the vertices of a graph such that no two adjacent vertices are colored with the same color. Similarly a proper edge coloring is defined as a coloring of the edges of the graph such that no two adjacent edges are colored with the same color. Hopcroft and Krishnamoorthy [7] introduced harmonious coloring of vertices.

Definition 1.2. [7] A harmonious coloring of a graph $G$ is an assignment of colors to the vertices of $G$ and the color of an edge is defined to be the unordered pair of colors assigned to its end vertices such that all edge colors are distinct. The harmonious coloring number is the least number of colors in such a coloring.

An enormous body of literature is available on harmonious coloring [8]. Hegde and Priya [6] have extended the definition of harmonious coloring to directed graphs.

Definition 1.3. Let $D$ be a directed graph with $n$ vertices and $m$ edges. A function $f : V(D) \rightarrow \{1, 2, \ldots, k\}$ where $k \leq n$ is said to be a harmonious coloring of $D$ if for any two edges $xy$ and $uv$ of $D$, the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If the pair $(i, i)$ is not assigned for any $i$, $1 \leq i \leq n$, then $f$ is called a proper harmonious coloring of $D$. The minimum $k$ for which $D$ admits a proper harmonious coloring is called the proper harmonious coloring number of $D$ and is denoted by $\overrightarrow{\chi}_h(D)$. 
Keith Edwards [9] has given an upper bound for the harmonious chromatic number of a general digraph and then showed that determining the exact value of the harmonious chromatic number is NP-hard for directed graphs of bounded degree.

In this paper we have further extended the concept of proper harmonious coloring number of a directed graph to all oriented graphs of an underlying undirected graph as follows:

Let \( D \) be an underlying undirected graph. Let \( O \) be an orientation of the edges of \( G \). Denote the directed graph with orientation \( O \) as \( G(O) \). Since there are \( 2|E| \) edges, there exist \( 2|E| \) different directed graphs say \( G(O_1), G(O_2), G(O_3), ..., G(O_{2|E|}) \). Let \( h_i \) denote the harmonious coloring number of \( G(O_i) \), \( 1 \leq i \leq 2|E| \). Define \( h = \min_i h_i \) as the harmonious coloring number of the oriented graph \( O(G) \) and denote it by \( \chi_h(D) \).

The following theorem gives a lower bound for \( \chi_h(D) \):

**Theorem 1.4.** [6] For any digraph \( D \), \( \chi_h(D) \geq \lceil \frac{1 + \sqrt{4m + 1}}{2} \rceil \) where \( m \) is the number of edges of \( D \).

Let \( G \) be an undirected graph. For any orientation \( O \) of the edges of \( G \), \( G(O) \) is a directed graph. Since Theorem 1.4 is true for \( G(O) \) for any arbitrary orientation \( O \), the following result holds good.

**Corollary 1.5.** Let \( G \) be any undirected graph. Then \( \chi_h(O(G)) \geq \lceil \frac{1 + \sqrt{4m + 1}}{2} \rceil \) where \( m \) is the number of edges of \( G \).

## 2 Main Results

In this section, we obtain a lower bound for proper harmonious coloring number of regular digraphs.

**Definition 2.1.** A digraph \( D \) is \( r \)-regular if the indegree of \( v \) = outdegree of \( v \) = \( r \) for every vertex \( v \) of \( D \).

**Theorem 2.2.** Let \( D \) be an \( r \)-regular digraph of order \( n \). Then \( \chi_h(D) \geq \lceil \frac{1 + \sqrt{1 + 4rn}}{2} \rceil \).

**Proof.** Let \( D \) be harmoniously colored with \( k \) colors. There exists at least one color class, say \( X \), that contains at least \( \frac{k}{r} \) vertices. For, if every color class contains less than \( \frac{k}{r} \) vertices, then the number of vertices in \( D \) is less than \( n \), which is a contradiction. Let \( N(X) \) denote the neighbourhood of the color class \( X \). Since \( D \) is regular of degree \( r \) and no two vertices in the same color class have a common in- or out- neighbour, it follows that there are at least \( r \left( \frac{k}{r} \right) \) vertices in \( N(X) \). Each of these vertices must be assigned a distinct color. Thus the total number of colors is \( k \geq |N(X)| + 1 \).

This implies \( k^2 - k - rn \geq 0 \). Therefore, \( (k - (\frac{1 + \sqrt{1 + 4rn}}{2})) (k - (\frac{1 - \sqrt{1 + 4rn}}{2})) \geq 0 \) but \( (k - (\frac{1 - \sqrt{1 + 4rn}}{2})) > 0 \) and hence \( (k - (\frac{1 + \sqrt{1 + 4rn}}{2})) \geq 0 \) Thus, \( \chi_h(D) \geq \lceil \frac{1 + \sqrt{1 + 4rn}}{2} \rceil \).

**Remark 2.3.** Theorem 2.2 can be derived as a corollary to Theorem 1.4.
Remark 2.4. When \( r = 1 \), \( \chi_h(D) \geq \lceil \frac{1+\sqrt{1+4n}}{2} \rceil \). But \( r = 1 \) implies that \( D \) is the unicycle \( C_n \) with \( n \) vertices. Also, Hegde and Priya [6] have already proved that for \( k = \lceil \frac{1+\sqrt{4n+1}}{2} \rceil \),

\[
\chi_h(C_n) = \begin{cases} 
  k + 1 & \text{if } n = k(k-1) - 1, \\
  k & \text{if } n = (k-1)(k-2) + 1, ..., k(k-1) - 2, k(k-1).
\end{cases}
\]

Let \( G \) be an undirected graph. Consider all orientations of edges of \( G \) such that \( G(O) \) is regular. Since Theorem 2.2 is true for all orientations of \( G \), we have the following result.

Corollary 2.5. Let \( G \) be an undirected \( 2r \)-regular graph. Then

\[
\chi_h(O(G)) \geq \left\lfloor \frac{1+\sqrt{1+4rn}}{2} \right\rfloor.
\]

3 Torus

Definition 3.1. An \( n \)-dimensional torus is defined as an interconnection structure that has \( k_0 \times k_1 \times \cdots \times k_{n-1} \) nodes where \( k_i \) is the number of nodes in \( i \)th dimension. Each node in the torus is identified by an \( n \)-coordinate vector \((x_0, x_1, \cdots, x_{n-1})\), where \( 0 \leq x_i \leq k_i - 1 \). Two nodes \((x_0, x_1, \cdots, x_{n-1})\) and \((y_0, y_1, \cdots, y_{n-1})\) are connected if and only if there exists an \( i \) such that \( x_i = (y_i \pm 1) \mod k_i \) and \( x_j = y_j \) for all \( j \neq i \).

Theorem 3.2. Let \( T \) be an undirected \( n \times n \) torus. Then \( \chi_h(O(T)) = k + 1 \) where

\[
k = \lceil \frac{1+\sqrt{1+8n^2}}{2} \rceil.
\]

Proof. By Theorem 2.2, \( \chi_h(O(T)) \geq k = \lceil \frac{1+\sqrt{1+8n^2}}{2} \rceil \). Let \( O \) be the orientation of \( T \) such that all the horizontal cycles are unidirected in the clockwise sense and all the vertical cycles are unidirected in the clockwise sense to obtain the orientation \( T(O) \) (See Figure 1).

![Figure 1: Orientation T(0) for 4 × 4 torus.](image)
A trail $\vec{W}$ of length $n(n + 1)$ in $\vec{K}_{k+1}$. Then the proper harmonious coloring number of $T(O)$ is equivalent to finding a closed trail $\vec{W}$ of length $n(n + 1)$ in $\vec{K}_{k+1}$ traversing through the edges of $\vec{K}_{k+1}$ at most once (See Figure 2) with the following conditions:

- In $\vec{W}$, the label of the vertex $v_2$ = the label of the vertex $v_{n(n-1)+n}$, the label of the vertex $v_3$ = the label of the vertex $v_{n(n-2)+(n-1)}$, the label of the vertex $v_4$ = the label of the vertex $v_{n(n-3)+(n-2)}$, ..., the label of the vertex $v_n$ = the label of the vertex $v_{n+2}$.

- In $\vec{W}$, the edges $(v_{n+2}, v_{2n+3})$; $(v_{2n+3}, v_{3n+4})$; $(v_{3n+4}, v_{4n+5})$; ...; $(v_2, v_1)$; $(v_{n+3}, v_{2n+4})$; $(v_{2n+4}, v_{3n+5})$; ...; $(v_2, v_1)$; $(v_{n+4}, v_{2n+5})$; $(v_{2n+5}, v_{3n+6})$; ...; $(v_{n+2}, v_3)$; ...; $(v_{2n}, v_{3n+1})$; $(v_{3n+1}, v_{4n+2})$; $(v_{4n+2}, v_{5n+3})$; ...; $(v_{n+1}, v_{n+1})$, $v_{n+1}$ do not exist. (See Figure 3)

- Also in $\vec{W}$, the edges $(v_1, v_{n+1})$; $(v_2, v_{n+3})$; $(v_3, v_{n+4})$; ...; $(v_{n-1}, v_{2n})$ do not exist. (See Figure 3)

Figure 2: (a) Oriented Torus $(4, 4)$; (b) Closed trail $\vec{W}$ in $\vec{K}_{6}$.

Figure 3: Oriented Torus $(n, n)$. 
There exists at least one such closed trail in $\overrightarrow{K}_{k+1}$ satisfying the above conditions. Further, $\overrightarrow{\chi}_h(O(T)) \neq k$ as the number of ordered pairs obtained from these $k$ colors are not sufficient to label any $n \times n$ grid $T(O)$ with distinct ordered pairs. Thus $\overrightarrow{\chi}_h(O(T)) = k + 1$.

4 Circulant Digraphs

The circulant is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [12]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [3]. It is also used in VLSI design and distributed computation [1, 2, 11]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [1]. Every circulant graph is a vertex transitive graph and a Cayley graph [13]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [1, 3].

The circulant digraph was proposed by Elspas and Turner [4].

Definition 4.1. A circulant digraph, denoted by $\overrightarrow{G}_n(S)$ where $S \subseteq \{1, 2, \cdots, n - 1\}$, $n \geq 2$, is defined as a digraph consisting of the vertex set $V = \{0, 1, \cdots, n - 1\}$ and the edge set $E = \{(i, j) : \text{there is } s \in S \text{ such that } j - i \equiv s \text{ (mod } n)\}$.

The digraph shown in Figure 4 is $\overrightarrow{G}_n(1, 3)$. It is clear that $\overrightarrow{G}_n(1)$ is the unicycle $\overrightarrow{C}_n$ and $\overrightarrow{G}_n(1, 2, \cdots, n - 1)$ is a complete digraph $\overrightarrow{K}_n$.

Theorem 4.2. Let $\overrightarrow{G}_n(1, 2)$ be a circulant digraph with $n$ vertices. Then for $k = \left\lceil \frac{1 + \sqrt{1 + 8n}}{2} \right\rceil$, $k \leq \overrightarrow{\chi}_h(\overrightarrow{G}_n(1, 2)) \leq k + 1$.

Proof. Clearly $\overrightarrow{G}_n(1, 2)$ is 2-regular digraph. Then by Theorem 2.2, $\overrightarrow{\chi}_h(\overrightarrow{G}_n(1, 2)) \geq k = \left\lceil \frac{1 + \sqrt{1 + 8n}}{2} \right\rceil$. 

Figure 4: Circulant digraph $\overrightarrow{G}_n(1, 3)$.

The directed cycle $\{0, 1, 2, \cdots, n - 1\}$ is called the outer cycle. All other cycles are called inner cycles.
Let $\vec{K}_{k+1}$ be a complete symmetric digraph with $k(k+1)$ edges. Label the vertices of $\vec{K}_{k+1}$ as $1, 2, ..., k+1$. Then there exists two cases:

**Case (i):** Let $n$ be odd.

Then in $\vec{G}_n(1, 2)$, there exists only one inner cycle. Let $v_1, v_2, v_3, ..., v_n, v_1$ be any closed trail $\vec{W}$ of length $n$ in $\vec{K}_{k+1}$ traversing through the edges of $\vec{K}_{k+1}$ at most once with the following conditions:

- If $(v_i, v_j)$ and $(v_j, v_k)$ are any two adjacent edges in $\vec{W}$, then the edge $(v_i, v_k)$ does not exist in $\vec{W}$ for any $i, j, k$, $1 \leq i \leq n$, $1 \leq j \leq n$, $1 \leq k \leq n$.
- The edge $(v_n, v_2)$ does not exist in the closed trail $\vec{W}$.

There exists at least one such closed trail in $\vec{K}_{k+1}$ satisfying the above conditions.

![Figure 5: Circulant digraph $\vec{G}_n(1, 2)$.](image)

**Case (ii):** Let $n$ be even.

Then in $\vec{G}_n(1, 2)$, there exist two inner cycles. Let $v = v_1, v_2, v_3, ..., v_{n-1}, v_n = v, v_{n+1}, v_{n+2}, ..., v_{2n-1}, v_{2n}, v_{2n+1}, v_{2n+2}, ..., v_{3n-1}, v_{3n}, v_{3n+1}, ..., v_{2n-1}, v_{2n+1}, v_{2n+2}$ be any closed trail $\vec{W}$ of length $2n$ in $\vec{K}_{k+1}$ traversing through the edges of $\vec{K}_{k+1}$ at most once with the following condition:

- In $\vec{W}$, the label of the vertex $v_1$ = the label of the vertex $v_n$ = the label of the vertex $v_{2n+1}$, the label of the vertex $v_2$ = the label of the vertex $v_{n+1}$, the label of the vertex $v_3$ = the label of the vertex $v_{n+2}$, ..., the label of the vertex $v_{n-1}$ = the label of the vertex $v_{2n-1}$, the label of the vertex $v_2$ = the label of the vertex $v_{n+1}$, the label of the vertex $v_4$ = the label of the vertex $v_{2n}$, the label of the vertex $v_5$ = the label of the vertex $v_{2n+1}$, ..., the label of the vertex $v_{2n}$ = the label of the vertex $v_{3n}$.

There exists at least one such closed trail in $\vec{K}_{k+1}$ satisfying the above condition.
The proper harmonious coloring number of $\vec{G}_n(1, 2)$ is equivalent to finding a closed trail $\vec{W}$ as defined above.

\section{Conclusion}

In this paper we have obtained the bound for proper harmonious coloring number of $G(O)$ when $G$ is a torus of order $n \times n$ and the circulant digraph $\vec{G}_n(1, 2)$. The problem of determining proper harmonious coloring number of oriented Torii of any order and circulant digraph $\vec{G}_n(S)$ when $S \subseteq \{1, 2, \cdots, n-1\}$, $n \geq 2$ is under investigation.

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\section*{References}


