A NOTE ON NORDHAUS-GADDUM-TYPE INEQUALITIES FOR THE AUTOMORPHIC $\mathcal{H}$-CHROMATIC INDEX OF GRAPHS *

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Abstract

The automorphic $H$-chromatic index of a graph $G$ is the minimum integer $m$ for which $G$ has a proper edge-coloring with $m$ colors which is preserved by a given automorphism group $H$ of $G$. We consider the sum and the product of the automorphic $H$-chromatic index of a graph and its complement. We prove upper and lower bounds in terms of the order of the graph when $H$ is chosen to be either a cyclic group of prime order or a group of order four.

1 Introduction

All graphs under consideration are simple. For graph terminology and notation we refer to [6]. Let $G = (V, E)$ be a graph of order $n$ with vertex set $V$ and edge set $E$. The complement $\bar{G}$ of a graph $G$ is the graph whose vertex set is that of $G$ and in which two vertices are adjacent if and only if they are not adjacent in $G$. Let $k \geq 2$ be an integer.

Following [7] we define a $k$-decomposition of a graph $G_0$ as a family $(G_1, G_2, \ldots, G_k)$ of spanning subgraphs of $G_0$ such that each edge of $G_0$ is contained in exactly one member of $(G_1, G_2, \ldots, G_k)$, see also [3]. We shall occasionally refer to the subgraphs $G_1, G_2, \ldots, G_k$ as being the “blocks” of the $k$-decomposition.

The following two problems can be formulated for an arbitrary graph parameter $P$:

(1) finding upper and lower bounds of the set
\[ \{P(G_1) + \cdots + P(G_k) : (G_1, G_2, \ldots, G_k) \text{ is a } k\text{-decomposition of } G_0\}; \]

(2) finding upper and lower bounds of the set
\[ \{P(G_1) \cdot P(G_2) \cdots P(G_k) : (G_1, G_2, \ldots, G_k) \text{ is a } k\text{-decomposition of } G_0\}. \]

The study of the above problems started in 1956 with the paper by Nordhaus and Gaddum [10] in the particular case $k = 2$, $G_0$ the complete graph $K_n$ of order $n$ and $P = \chi$ the chromatic number. Nordhaus and Gaddum gave answers to problems (1) and (2) in terms of the order $n$ of $G_0 = K_n$. Only 10 years later Vizing in [11] solved the same problems for another graph parameter, namely the chromatic index $P = \chi'$.

Theorem 1.1. [11] For an arbitrary graph $G$ of order $n$ the following inequalities hold:
\[ 2 \left\lfloor \frac{n + 1}{2} \right\rfloor - 1 \leq \chi'(G) + \chi'(\bar{G}) \leq n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor, \]
0 \leq \chi'(G)\chi'(\overline{G}) \leq (p - 1) \left( 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \right).

Theorem 1.1 was independently proved by Alavi and Behzard in [1] and by Capobianco and Molluzzo in [5]. This type of result, called Nordhaus-Gaddum-type inequalities, have been studied for several different graph parameters. We refer to [2] for a recent survey on Nordhaus-Gaddum-type inequalities. In particular in [2, Sec.3.6] Nordhaus-Gaddum-type inequalities can be found for several chromatic graph parameters, for example, the total chromatic number.

In this paper we consider Nordhaus-Gaddum-type inequalities for a particular chromatic graph parameter: the automorphic chromatic index. Let \( \phi : E \to C \) be an edge-coloring of a graph \( G \) with color set \( C \). An automorphism \( \sigma \) of \( G \) preserves \( \phi \) if there exists a permutation \( a \) of the color-set \( C \) such that the relation \( \phi \sigma(e) = a\phi(e) \) holds for each \( e \in E \). Denote by \( \text{Aut}(G) \) the full automorphism group of a graph \( G \) and by \( \mathcal{H} \) a given subgroup of \( \text{Aut}(G) \). The automorphic \( \mathcal{H} \)-chromatic index of \( G \), as defined in [8] and denoted by \( \chi'_\mathcal{H}(G) \), is the minimum integer \( m \) for which \( G \) has a proper edge-coloring with \( m \) colors preserved by each automorphism of the subgroup \( \mathcal{H} \).

Upper bounds for \( \chi'_\mathcal{H}(G) \) in terms of the chromatic index \( \chi'(G) \) are established in [8] and [9] when \( \mathcal{H} \) is either a cyclic group of prime order or a group of order four. We recall these results which shall be used in Section 3.

**Proposition 1.2.** [8] Let \( G \) be a graph with chromatic index \( \chi'(G) \) and assume that \( \mathcal{H} \) is cyclic of order 2. Then the inequality holds:

\[
\chi'_\mathcal{H}(G) \leq \chi'(G) + 2 \left\lfloor \frac{\chi'(G)}{2} \right\rfloor.
\]

Let \( \sigma \) be an automorphism of \( G \) of odd prime order \( p \). A \( \sigma \)-cycle is a cycle of \( G \) of length \( p \) which is preserved by \( \sigma \) while none of its vertices is fixed by \( \sigma \).

**Proposition 1.3.** [8] Let \( G \) be a graph with chromatic index \( \chi'(G) \) and assume that \( \mathcal{H} \) is cyclic of odd prime order \( p \) and generated by \( \sigma \). Then the inequality holds:

\[
\chi'_\mathcal{H}(G) \leq \chi'(G) + p \left\lfloor \frac{\chi'(G)}{p} \right\rfloor
\]

provided that \( G \) has either no \( \sigma \)-cycles or maximum degree not divisible by \( p \).

**Proposition 1.4.** [9] Let \( G \) be a graph with chromatic index \( \chi'(G) \) and assume that \( \mathcal{H} \) is the Klein group. Then the inequality holds:

\[
\chi'_\mathcal{H}(G) \leq \chi'(G) + 6 \left\lfloor \frac{\chi'(G)}{2} \right\rfloor + 4 \left\lfloor \frac{\chi'(G)}{4} \right\rfloor.
\]
**Proposition 1.5.** [9] Let $G$ be a graph with chromatic index $\chi'(G)$ and assume that $\mathcal{H}$ is cyclic of order four. Then the inequality holds:

$$\chi'_\mathcal{H}(G) \leq \chi'(G) + 2 \left\lceil \frac{\chi'(G)}{2} \right\rceil + 4 \left\lceil \frac{\chi'(G)}{2} \right\rceil + 4.$$

All the above described bounds, with the exception of one, are best possible (see [8] and [9]). In this note the main purpose is to find Nordhaus-Gaddum-type inequalities for the $\mathcal{H}$-automorphic chromatic index of a graph $G$ with $\mathcal{H}$ either a cyclic group of prime order or a cyclic group of order four.

## 2 Some general bounds

In this section a result is shown in analogy to [3, Theorem 4.9 p. 27].

**Definition 2.1.** Let $(G_1, G_2, \ldots, G_k)$ be a $k$-decomposition of a graph $G_0$ and $\mathcal{H}$ a subgroup of $\text{Aut}(G_0)$. The $k$-decomposition $(G_1, G_2, \ldots, G_k)$ is said to be blockwise fixed by $\mathcal{H}$ if $G^h_i = G_i$ holds for $h \in \mathcal{H}$ and $i = 1, 2, \ldots, k$.

In what follows we shall omit the word blockwise and we denote by $H_i$ the automorphism group of $G_i$ induced by $\mathcal{H}$ on $G_i$ for $i = 1, 2, \ldots, k$.

**Lemma 2.2.** Let $(G_1, G_2, \ldots, G_k)$ be a $k$-decomposition of $G_0$ which is fixed by $\mathcal{H}$, $\mathcal{H} \leq \text{Aut}(G_0)$. Then,

$$\chi'_\mathcal{H}(G_0) \leq \chi'_{\mathcal{H}_1}(G_1) + \chi'_{\mathcal{H}_2}(G_2) + \cdots + \chi'_{\mathcal{H}_k}(G_k).$$

**Proof.** For $i = 1, 2, \ldots, k$, let $m_i = \chi'_{\mathcal{H}_i}(G_i)$ be and let $\phi_i$ be an edge-coloring of $G_i$ with colors $C_i$ preserved by $\mathcal{H}_i$ such that $|C_i| = m_i$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Each edge of $G_0$ is contained in exactly one member of $(G_1, G_2, \ldots, G_k)$. If an edge $e$ of $G_0$ belongs to $G_i$ then we color it with $\phi_i(e)$. Therefore an edge-coloring of $G$ with $(m_1 + m_2 + \cdots + m_k)$ colors is obtained which is preserved by $\mathcal{H}$ by construction. Obviously we have the following inequality

$$\chi'_\mathcal{H}(G_0) \leq m_1 + m_2 + \cdots + m_k = \chi'_{\mathcal{H}_1}(G_1) + \chi'_{\mathcal{H}_2}(G_2) + \cdots + \chi'_{\mathcal{H}_k}(G_k),$$

and the statement follows. \(\square\)

**Proposition 2.3.** Let $n$ and $k$ be positive integers and $(G_1, G_2, \ldots, G_k)$ be a $k$-decomposition of $K_n$ which is fixed by $\mathcal{H}$, $\mathcal{H} \leq \text{Aut}(K_n)$. Then,

$$2 \left\lceil \frac{n + 1}{2} \right\rceil - 1 \leq \chi'_{\mathcal{H}_1}(G_1) + \chi'_{\mathcal{H}_2}(G_2) + \cdots + \chi'_{\mathcal{H}_k}(G_k).$$
Proof. Since $\chi'(K_n) = 2\left\lceil \frac{n+1}{2} \right\rceil - 1$ and $\chi'(K_n) \leq \chi'_H(K_n)$, Lemma 2.2 implies the statement. This lower bound is the best possible: if $H$ is the identity group and $G_1 = K_n$, then $\chi'_H(G_i)$ coincides with $\chi'(G_i)$. Hence, we get $\chi'_H(G_1) = 2\left\lceil \frac{n+1}{2} \right\rceil - 1$ and $\chi'_H(G_i) = 0$ for $i \neq 1$. □

3 Some Nordhaus-Gaddum-type inequalities for the $H$-automorphic chromatic index

In this section $G_0$ will be the complete graph $K_n$ of order $n$ and a 2-decomposition of $K_n$ will be denoted by $(G, \bar{G})$ where $\bar{G}$ is the complement of $G$. The automorphism group of a graph $G$ coincides with the automorphism group of the complement of $G$, [4, Theorem 1.1 p. 139], therefore, if $H \leq \text{Aut}(G)$ both $\chi'_H(G)$ and $\chi'_H(\bar{G})$ can be studied simultaneously. Proposition 2.3 in the particular case $k = 2$ implies the following:

Lemma 3.1. For an arbitrary graph $G$ of order $n$ with $H \leq \text{Aut}(G)$ the following inequality holds:

$$2\left\lceil \frac{n+1}{2} \right\rceil - 1 \leq \chi'_H(G) + \chi'_H(\bar{G}).$$

The above bound is best possible as shown in Proposition 2.3.

Lemma 3.2. Let $G$ be a graph of order $n$. Then,

$$\left\lceil \frac{\chi'(G)}{r} \right\rceil + \left\lceil \frac{\chi'(\bar{G})}{r} \right\rceil \leq \frac{1}{r} \left( n + 2 \left\lceil \frac{(n-2)}{2} \right\rceil \right) + 2$$

where $r$ is an integer greater than or equal to 1.

Proof. The statement follows from the following inequalities

$$\left\lceil \frac{\chi'(G)}{r} \right\rceil + \left\lceil \frac{\chi'(\bar{G})}{r} \right\rceil \leq \frac{1}{r} (\chi'(G) + \chi'(\bar{G})) + 2 \leq \frac{1}{r} \left( n + 2 \left\lceil \frac{(n-2)}{2} \right\rceil \right) + 2$$

where the last one is obtained from Theorem 1.1. □

Proposition 3.3. Let $G$ be a graph of order $n$ with $H \leq \text{Aut}(G)$ and assume that $H$ is cyclic of order 2. Then the following inequality holds:

$$\chi'_H(G) + \chi'_H(\bar{G}) \leq 2 \left( n + 2 \left\lceil \frac{(n-2)}{2} \right\rceil \right) + 4.$$

Proof. We get

$$\chi'_H(G) + \chi'_H(\bar{G}) \leq \chi'(G) + \chi'(\bar{G}) + 2 \left\lceil \frac{\chi'(G)}{2} \right\rceil + 2 \left\lceil \frac{\chi'(\bar{G})}{2} \right\rceil$$
Nordhaus-Gaddum-type inequalities

\[
\leq n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor + 2 \left( \frac{1}{2} \left( n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \right) + 2 \right) \leq 2 \left( n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \right) + 4
\]

where the first of the above relations is obtained from Proposition 1.2, while the second one from Theorem 1.1 and Lemma 3.2.

Proposition 3.4. Let \( G \) be a graph of order \( n \), with \( \mathcal{H} \leq \text{Aut}(G) \). Assume that \( \mathcal{H} \) is cyclic of odd prime order \( p \) and generated by \( \sigma \). The following inequalities hold:

\[
p \leq \chi'_\mathcal{H}(G) + \chi'_\mathcal{H}(\bar{G}) \leq 2 \left( n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \right) + 2p
\]

provided that \( G \) does not contain \( \sigma \)-cycles and \( \bar{G} \) has maximum degree not divisible by \( p \).

Proof. From Proposition 1.3, Theorem 1.1 and Lemma 3.2 we have

\[
\chi'_\mathcal{H}(G) + \chi'_\mathcal{H}(\bar{G}) \leq \chi'(G) + \chi'(\bar{G}) + p \left\lfloor \frac{\chi'(G)}{p} \right\rfloor + p \left\lfloor \frac{\chi'(\bar{G})}{p} \right\rfloor
\]

\[
\leq n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor + p \left( \frac{1}{p} \left( n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \right) + 2 \right)
\]

\[
\leq 2 \left( n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \right) + 2p.
\]

Hence, the upper bound is proved.

In \( G \) there exist \( p \) vertices \( v_0, v_1, \ldots, v_{p-1} \) such that \( \sigma(v_i) = v_{i+1} \), where the indices are taken modulo \( p \). Since \( G \) has no \( \sigma \)-cycles, then \( C = v_0v_1 \ldots v_{p-1}v_0 \) is a cycle in \( G \). In order to have an edge-coloring of \( G \) preserved by \( \sigma \), the edges of the cycle \( C \) must be colored with \( p \) different colors. Therefore \( \chi'_\mathcal{H}(\bar{G}) \geq p \) and the lower bound is shown.

We now prove that the lower bound is best possible. Using the cyclic group \( \mathbb{Z}_p \) of rotations of the \( p \)-gon, let \( \bar{G} = K_p \) be with \( V(\bar{G}) = \mathbb{Z}_p = \{0, 1, \ldots, p-1\} \), \( \sigma(i) = i + 1 \), (the indices are taken modulo \( p \)) and \( G \) the null graph of order \( p \). The “standard” edge-coloring of \( K_p \) with \( p \) colors and with all the edges of the near 1-factor \( F_i = \{ \{i + j, i - j\} : j \in \mathbb{Z}_p \setminus \{0\} \} \) colored by the color \( i \) \( (i = 0, 1, \ldots, p-1) \), is obviously preserved by \( \sigma \). Hence, \( p = \chi'(\bar{G}) \leq \chi'_\mathcal{H}(\bar{G}) \leq p \). Therefore, \( \chi'_\mathcal{H}(\bar{G}) = p \) and the lower bound is attained.

In analogy to the above propositions, Proposition 1.4 and Proposition 1.5 imply the following propositions.

Proposition 3.5. Let \( G \) be a graph of order \( n \) with \( \mathcal{H} \leq \text{Aut}(G) \). Assume that \( \mathcal{H} \) is the Klein group, then the following inequality holds:

\[
\chi'_\mathcal{H}(G) + \chi'_\mathcal{H}(\bar{G}) \leq 5 \left( n + 2 \left\lfloor \frac{n - 2}{2} \right\rfloor \right) + 20.
\]
Proposition 3.6. Let $G$ be a graph of order $n$ with $\mathcal{H} \leq \text{Aut}(G)$. Assume that $\mathcal{H}$ is cyclic of order 4, then the following inequality holds:

$$\chi'_\mathcal{H}(G) + \chi'_\mathcal{H}(\bar{G}) \leq 4 \left( n + 2 \left\lfloor \frac{n-2}{2} \right\rfloor \right) + 12.$$ 

Note that if $G = K_n$ then $\bar{G}$ is the null graph with $\chi'_\mathcal{H}(\bar{G}) = 0$, hence $0 \leq \chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G})$.

Since

$$\chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G}) = \left( \sqrt{\chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G})} \right)^2 \leq \left( \frac{\chi'_\mathcal{H}(G) + \chi'_\mathcal{H}(\bar{G})}{2} \right)^2$$

then by Propositions 3.3, 3.4, 3.5 and 3.6 we obtain the following:

Proposition 3.7. Let $G$ be a graph of order $n$ with $\mathcal{H} \leq \text{Aut}(G)$. Assume $\mathcal{H}$ cyclic of order 2, then the following inequalities hold:

$$0 \leq \chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G}) \leq \left( n + 2 \left\lfloor \frac{n-2}{2} \right\rfloor + 2 \right)^2.$$

Proposition 3.8. Let $G$ be a graph of order $n$ with $\mathcal{H} \leq \text{Aut}(G)$. Assume that $\mathcal{H}$ is cyclic of odd prime order $p$ and generated by $\sigma$. Then the following inequalities hold:

$$0 \leq \chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G}) \leq \left( n + 2 \left\lfloor \frac{(n-2)}{2} \right\rfloor + p \right)^2.$$

provided that $G$ does not contain $\sigma$-cycles and the maximum degree of $\bar{G}$ is not divisible by $p$.

Proposition 3.9. Let $G$ be a graph of order $n$ with $\mathcal{H} \leq \text{Aut}(G)$. Assume that $\mathcal{H}$ is the Klein group, then the following inequalities hold:

$$0 \leq \chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G}) \leq \left( \frac{5}{2} \left( n + 2 \left\lfloor \frac{(n-2)}{2} \right\rfloor \right) + 10 \right)^2.$$

Proposition 3.10. Let $G$ be a graph of order $n$ with $\mathcal{H} \leq \text{Aut}(G)$. Assume that $\mathcal{H}$ is cyclic of order 4, then the following inequalities hold:

$$0 \leq \chi'_\mathcal{H}(G)\chi'_\mathcal{H}(\bar{G}) \leq \left( 2n + 4 \left\lfloor \frac{(n-2)}{2} \right\rfloor + 6 \right)^2.$$

It remains an open problem to verify if some of the above bounds are sharp.

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